MCEN 7221 – Turbulence

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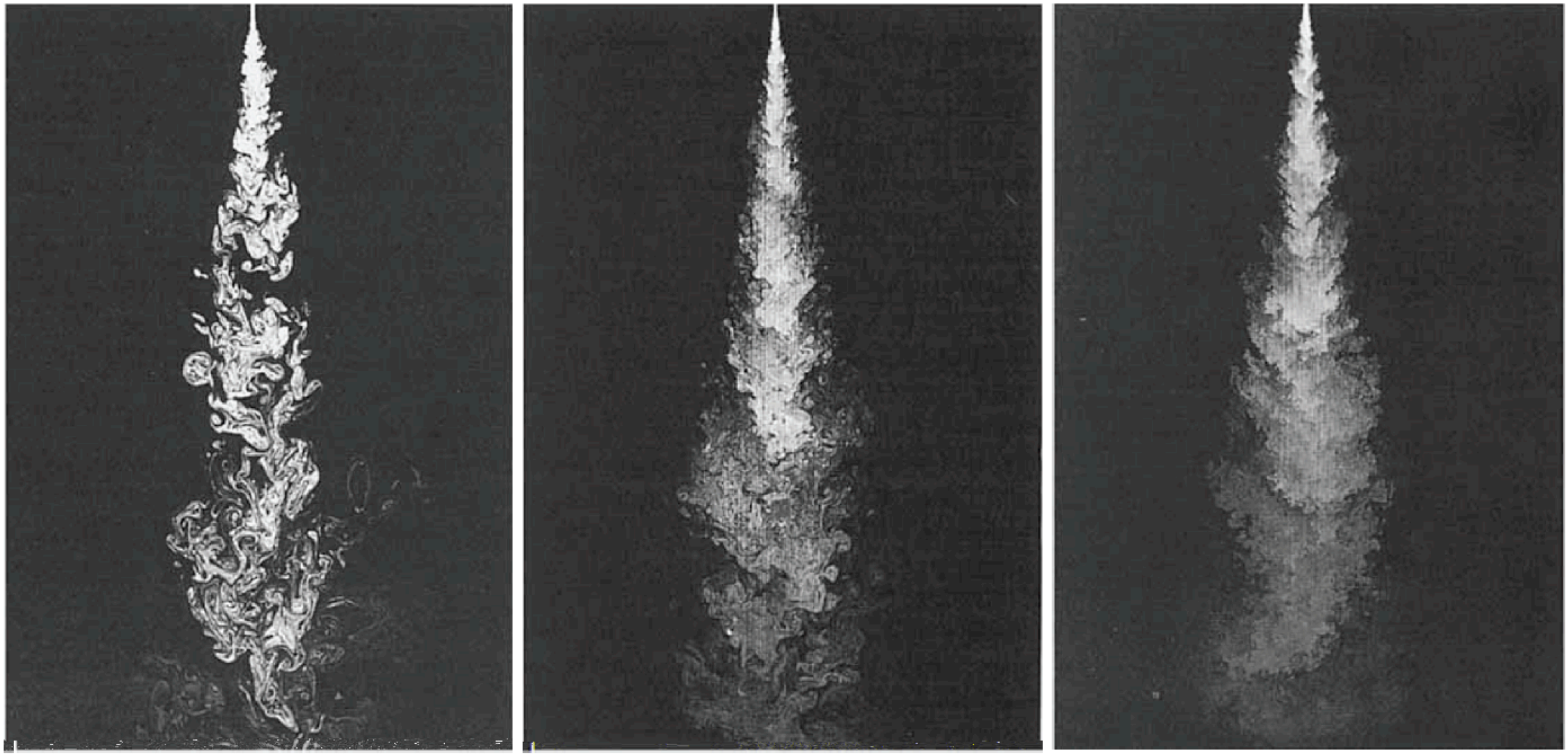


Re = 1,500 Re = 5,000 Re = 20,000

Course Topics

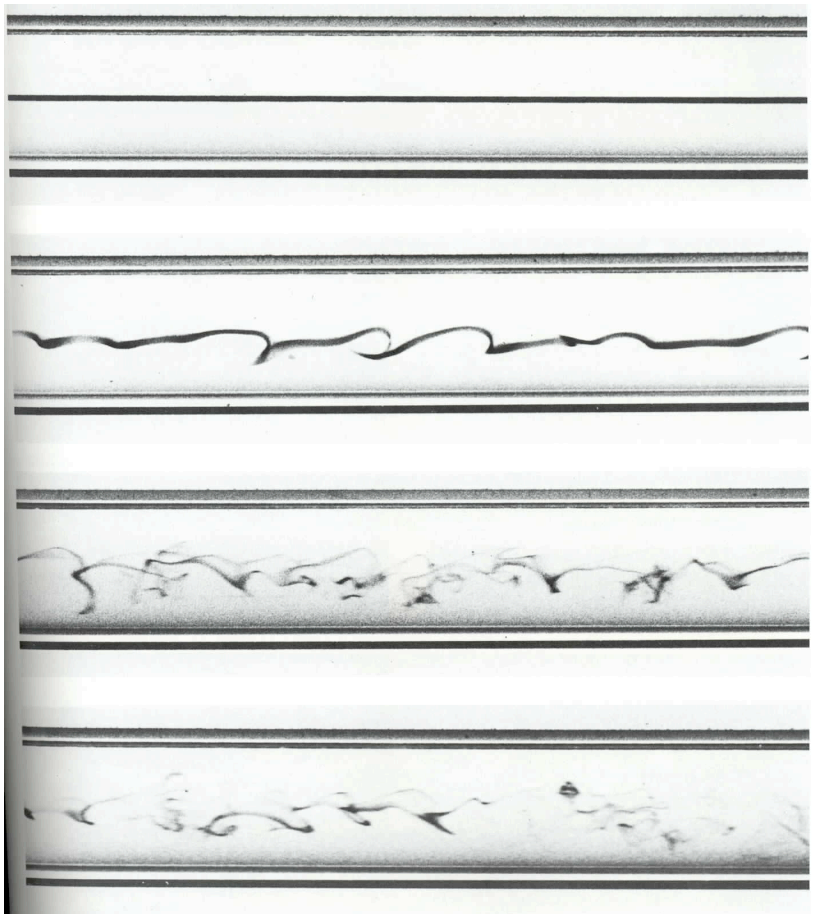
- Introduction to turbulence, equations of fluid motion, cartesian tensors
- Introduction to flow stability theory
- Statistical description of turbulence, mean flow equations
- Round jet, shear flows, grid turbulence
- Energy cascade, energy transfer, Kolmogorov theory
- Velocity spectra
- Channel and pipe flows
- Turbulent boundary layers
- Turbulence modeling: viscosity models, Reynolds stress models
- Large-eddy simulations and direct numerical simulations

Photo/movie contest

Introduction to Turbulence

- What is Turbulence?
  - Random, chaotic
  - Fully 3D
  - Has vorticity, high mixing, many scales
  - At large Re
- Why is it important?
  - Many flows of technical relevance are turbulent
  - Turbulence often desired in technical processes to accelerate mixing

Reynolds’ Pipe Flow Experiment (1883)



103. **Repetition of Reynolds’ dye experiment.** Osborne Reynolds’ celebrated 1883 investigation of stability of flow in a tube was documented by sketches rather than photography. However the original apparatus has survived at the University of Manchester. Using it a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs. In laminar flow a filament of colored water introduced at a bell-shaped entry extends undisturbed the whole length of the glass tube. Transition is seen in the second of the photographs as the speed is increased; and the last two photographs show fully turbulent flow. Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds.

Observation:

- For sufficiently high Re, flow becomes turbulent

Introduction to Turbulence

Governing Equations

Navier-Stokes equations describe turbulence, but the solution of realistic problems is too expensive

Momentum Equation

$$\frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 U_j}{\partial x_i^2}$$

Consider jet and non-dimensionalize with jet diameter D and jet exit bulk velocity U_b

$$\hat{U}_j = \frac{U_j}{U_b} \qquad \hat{x}_j = \frac{x_j}{D} \qquad \hat{p} = \frac{p}{\rho U_b^2} \qquad \hat{t} = \frac{tU_b}{D}$$

$$\Rightarrow \quad \frac{\partial \hat{U}_j}{\partial \hat{t}} + \hat{U}_i \frac{\partial \hat{U}_j}{\partial \hat{x}_i} = -\frac{\partial \hat{P}}{\partial \hat{x}_j} + \underbrace{\frac{\nu}{U_b D}}_{1/\mathrm{Re}} \frac{\partial^2 \hat{U}_j}{\partial \hat{x}_i^2}$$

For turbulent flows, viscous term important even if Reynolds number becomes very large.

Introduction to Turbulence

Similarity

Unknowns: $\hat{U}_j, \hat{p} = f(\hat{x}_j, \hat{t}, U_b, D, \nu)$

Buckingham Π-Theorem:

#Nondimensional Groups = #Parameters - #Units

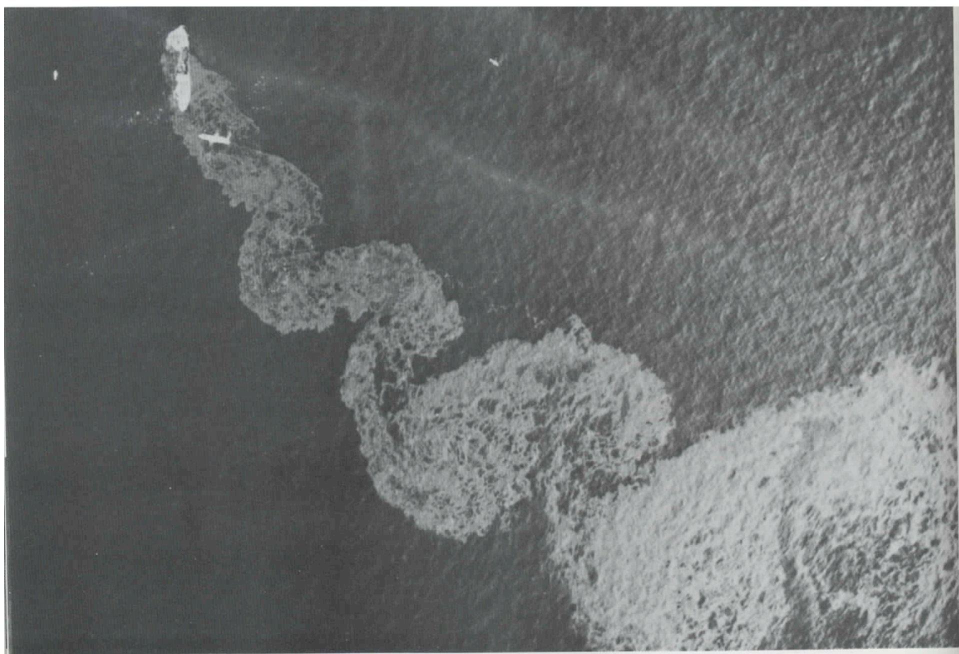
= 7 - 2 = 5

$\Rightarrow$ Five nondimensional groups: $\hat{x}_j, \hat{t}, \mathrm{Re}$

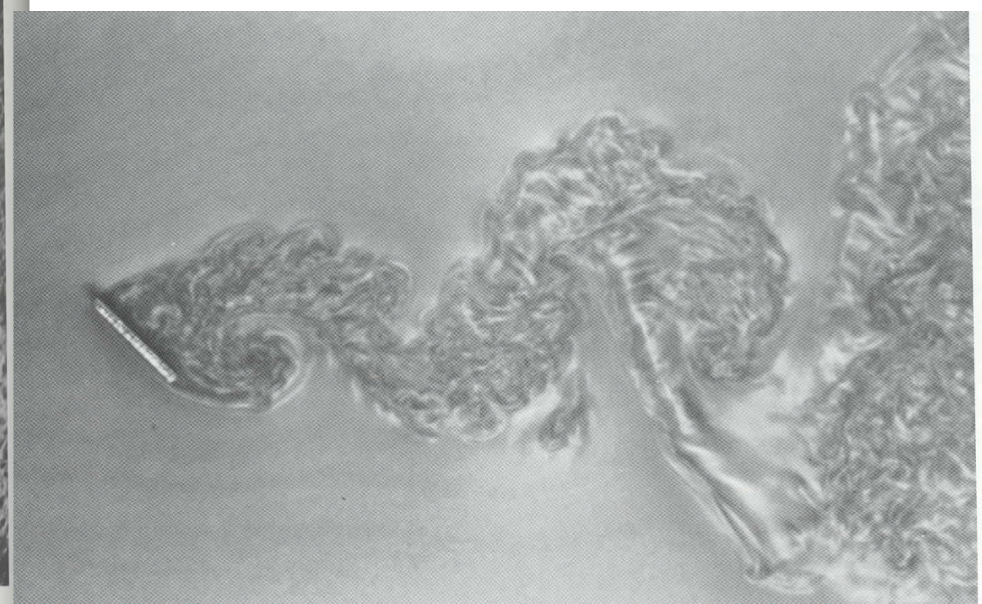
For large Re, solution becomes independent of Re

Wake of a Grounded Ship at Re = 10^7 and Re = 4300

For large Re, solution becomes independent of Re



173. Wake of a grounded tankship. The tanker *Argo Merchant* went aground on the Nantucket shoals in 1976. Leaking crude oil shows that she happened to be inclined at about 45° to the current. Although the Reynolds number is approximately 10^7, the wake pattern is remarkably similar to that in the photograph at the top of the page. *NASA photograph, courtesy of O. M. Griffin, Naval Research Laboratory.*



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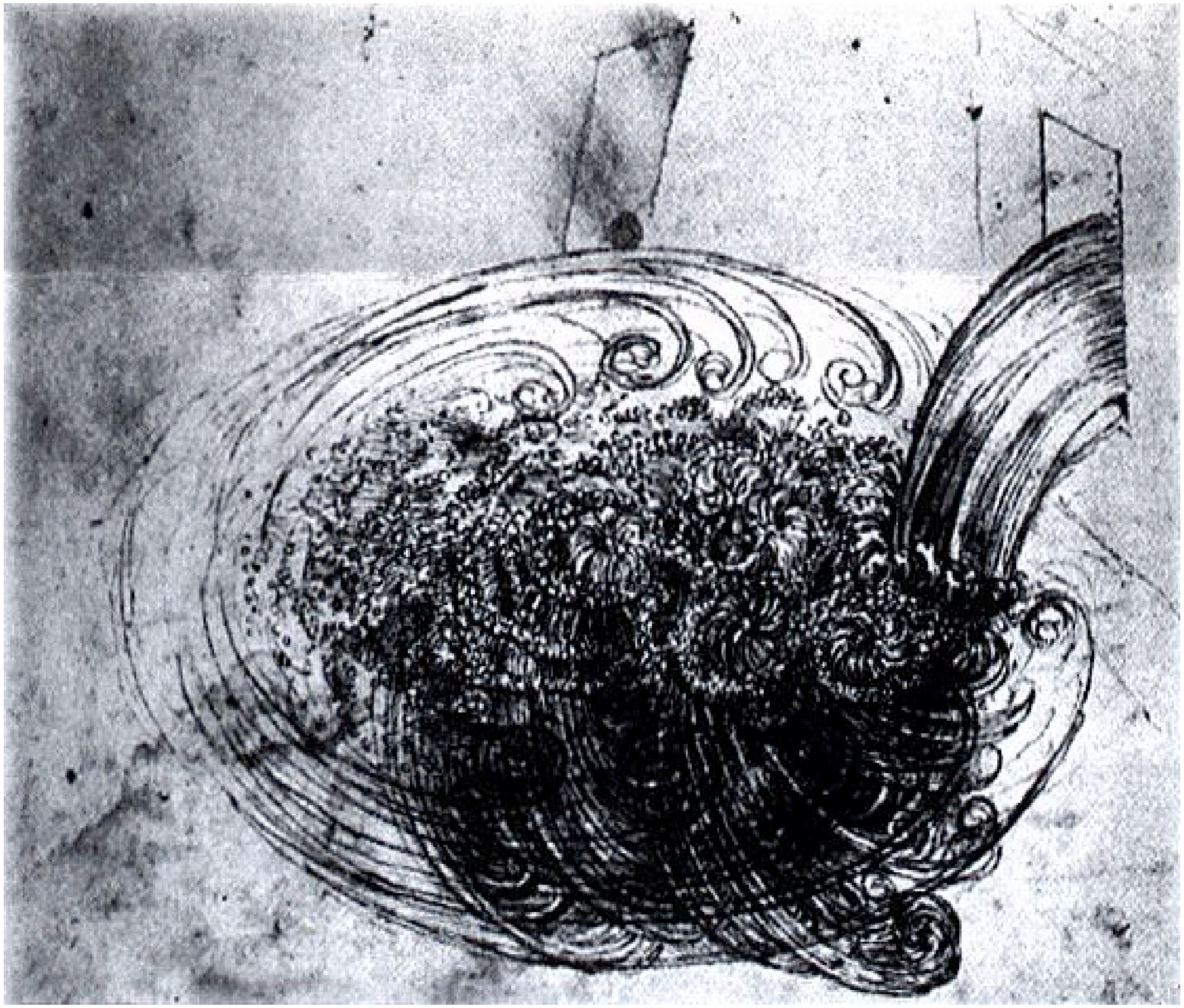
Turbulent Scales

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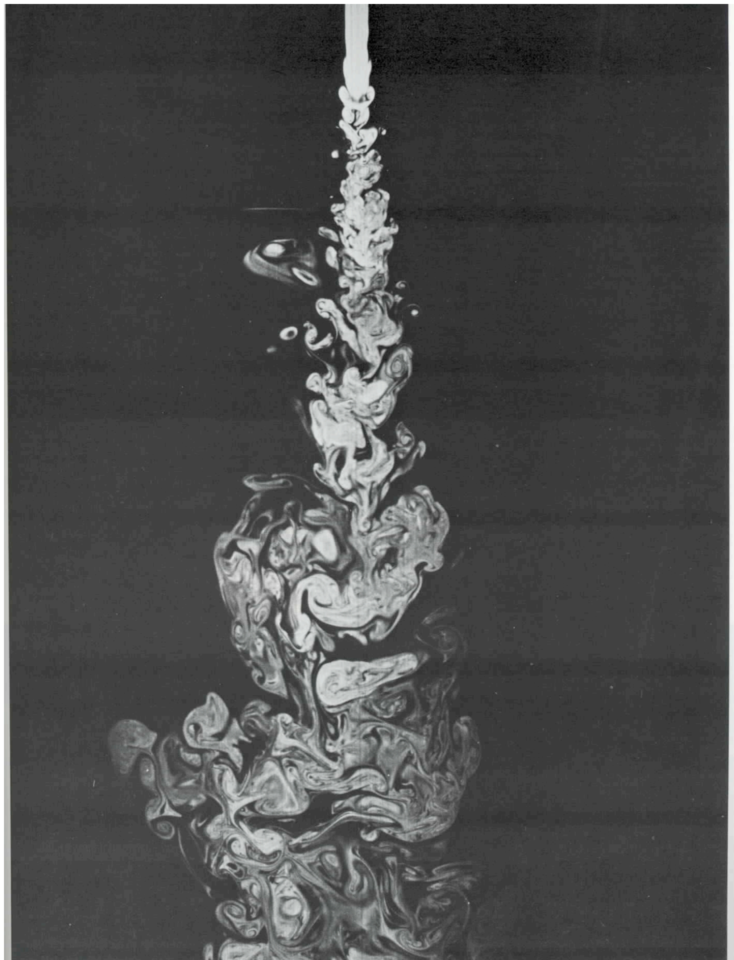
Scales of Turbulence

Turbulent Jet at Re = 2300



166. Turbulent water jet. Laser-induced fluorescence shows the concentration of jet fluid in the plane of symmetry of an axisymmetric jet of water directed downward into water. The Reynolds number is approximately 2300. The spatial resolution is adequate to resolve the Kolmogorov scale in the downstream half of the photograph. *Dimotakis, Lye & Papantoniou 1981*

- Identify turbulent length scale as the characteristic size of a pocket of fluid of coherent motion
- Then, the flow velocity of that pocket relative to the surroundings is the characteristic velocity scale at that length scale
- Associated with this velocity scale is the kinetic energy of that scale
- Use turbulent eddy as a “crutch”

Turbulent Eddies

Example: Heat transfer in internal combustion engine from Lumley: Engines, An Introduction

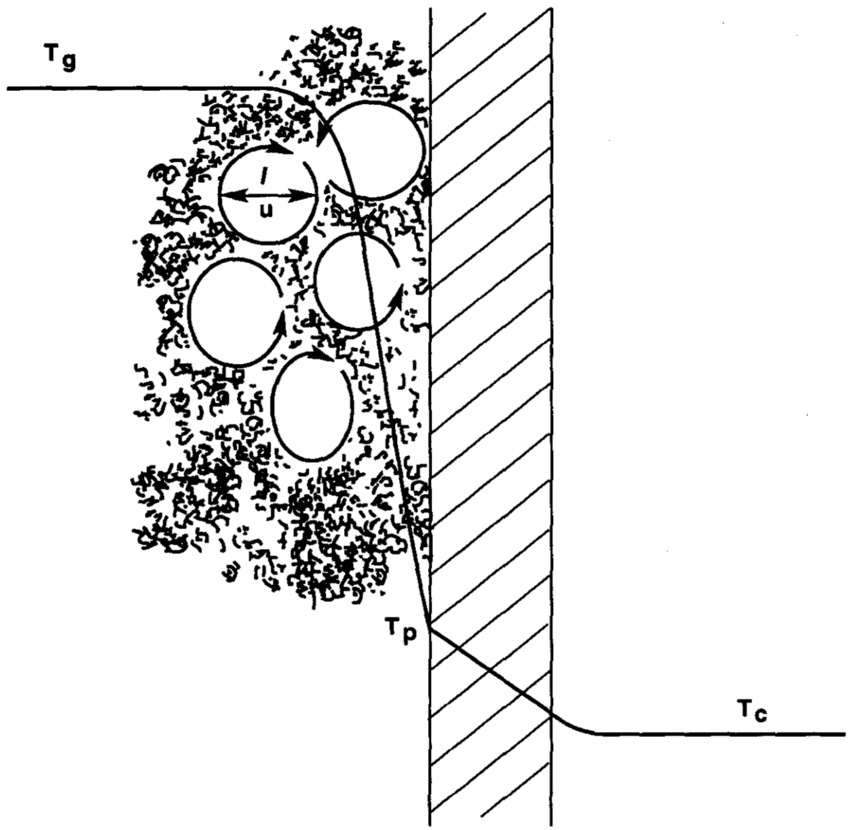
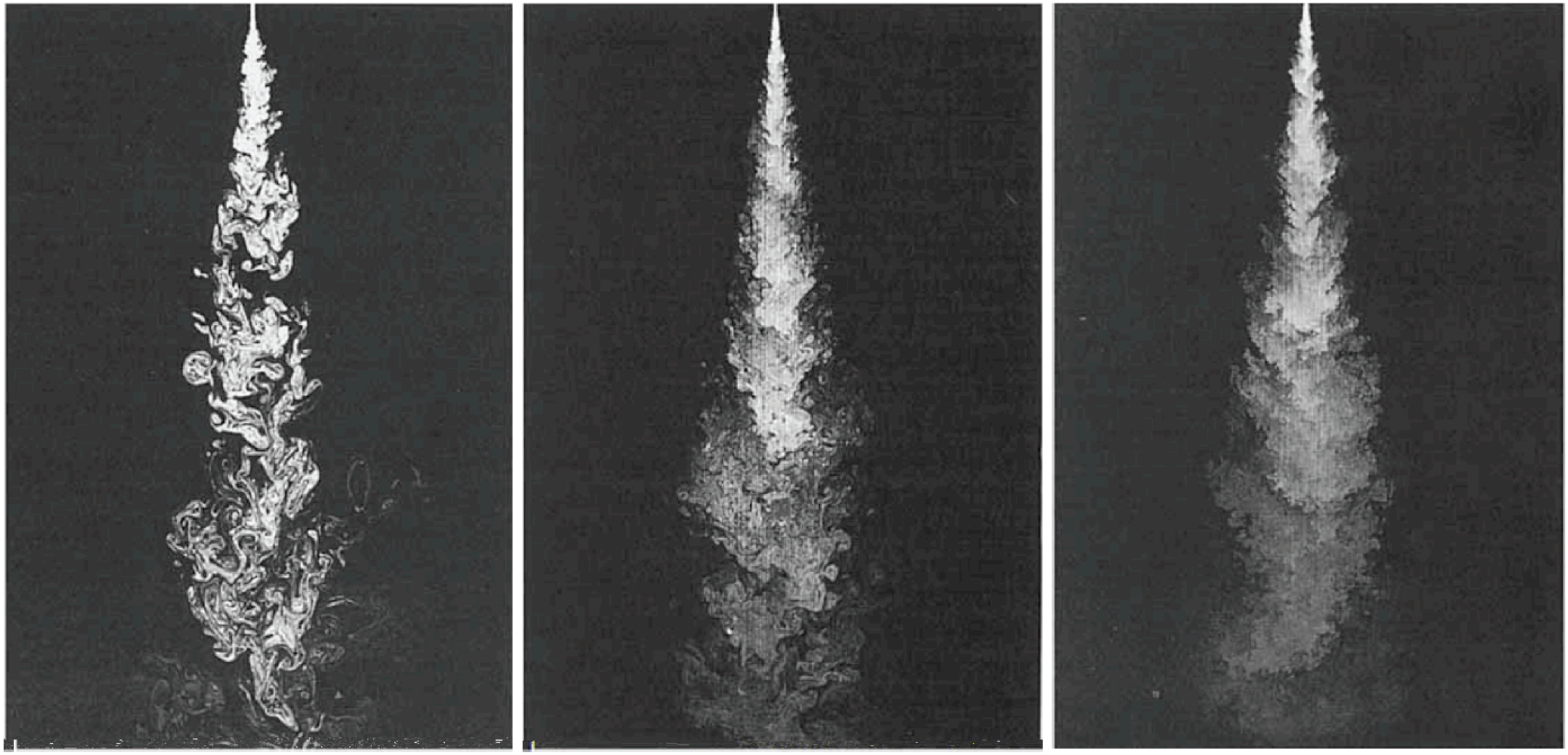


Figure 3.4. Qualitative picture of turbulent heat transfer in the cylinder.

Turbulent Jets at Different Reynolds Number



Re = 1,500 Re = 5,000 Re = 20,000

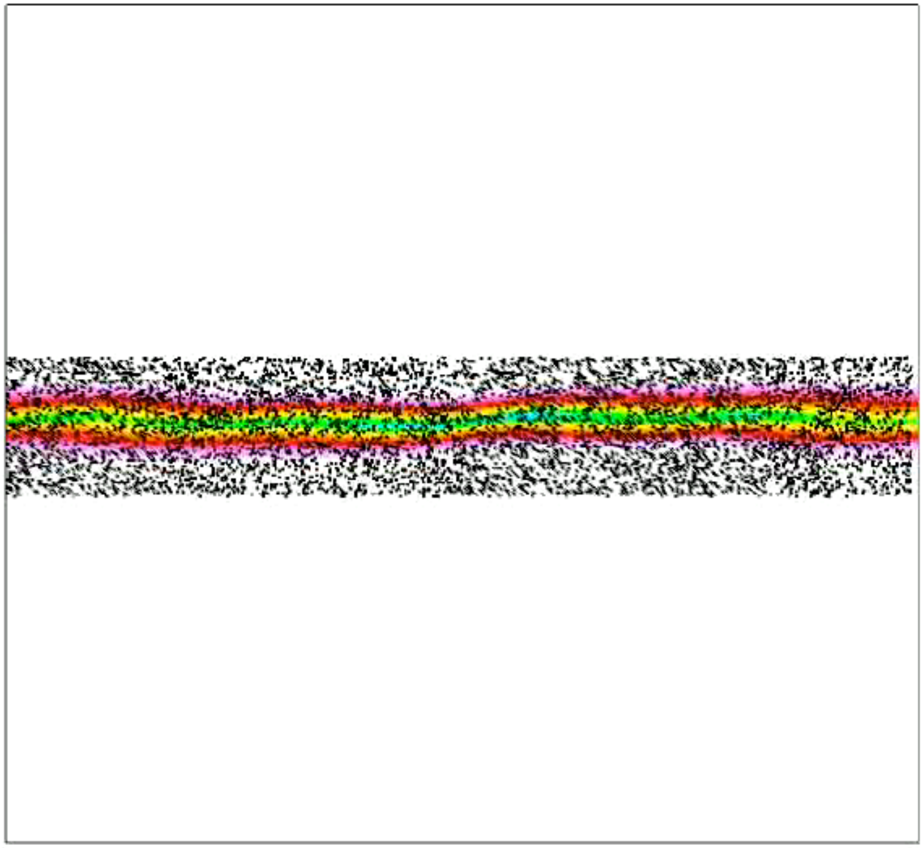
Scaling Arguments

- Scaling arguments very important in turbulence theory
- What is scaling argument?
  - Characterized by proportionality
  - Example:
    - Size of large turbulent structures scales with jet width
    - What is a scale characterizing the large turbulent scales?
      - Jet width
      - Jet half width
      - Distance from nozzle

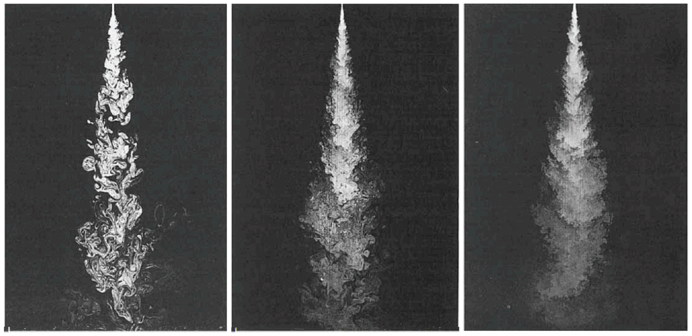
Kinetic Energy

- Turbulence often characterized by kinetic energy
- Turbulent kinetic energy refers to velocity differences
- For laminar or turbulent cases, geometry creates kinetic energy (velocity differences) at system scale
- Examples
  - Wake, jet, or shear layer
- Turbulence extracts kinetic energy at scale of geometry

Example: Turbulent Shear Layer



Turbulent Jets at Different Reynolds Number



Re = 1,500   Re = 5,000   Re = 20,000

Observation:

2. Turbulent flows have structures of different length scales

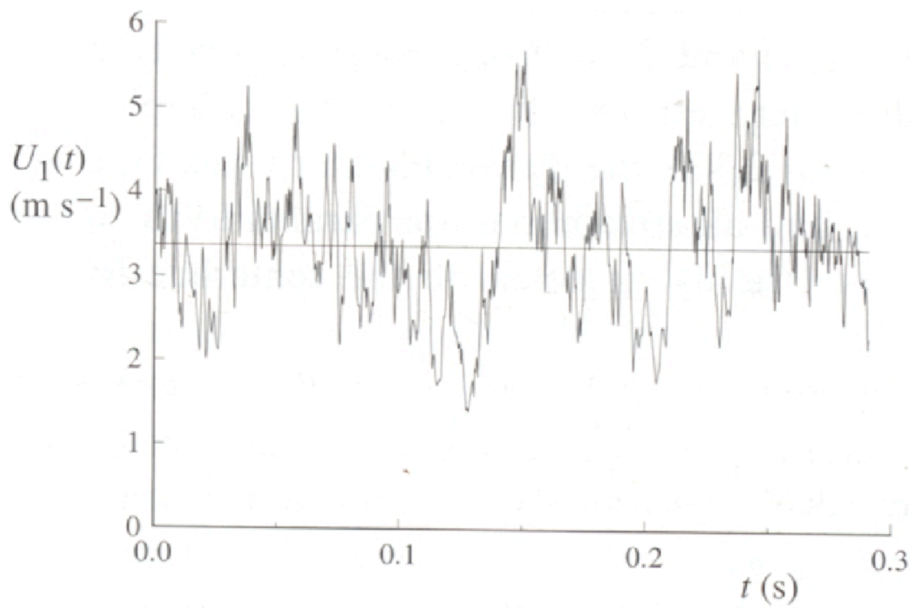
3. Smaller scales depend on Re

4. Large-scale features independent of Reynolds number

Kinetic energy of large scale turbulent motions extracted from mean shear

Turbulence produced at large scales

Velocity Time History on the Centerline of a Turbulent Jet



Observation:

4. Smaller scales lead to larger gradients

$\Rightarrow$ Viscous term becomes important

Introduction to Turbulence

A Few Conclusions for Turbulent Flows:

- Momentum equation

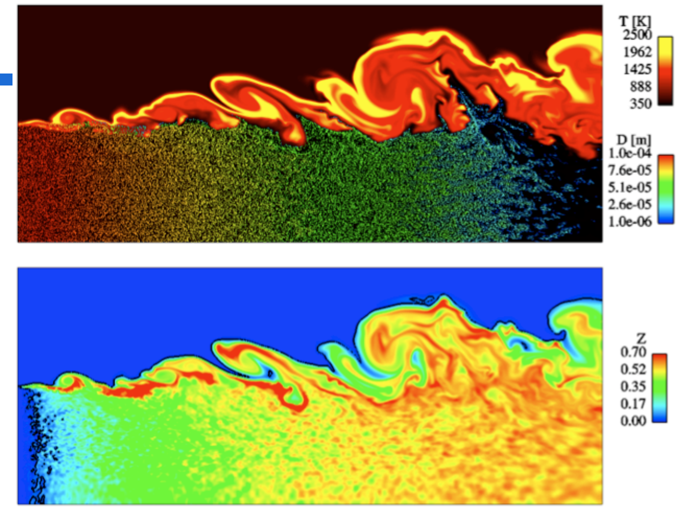
$$\frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 U_j}{\partial x_i^2}$$

- For large Re number laminar flow, viscous term disappears
- No damping of disturbances
- Instabilities lead to turbulent flow
- Turbulence produced at large scales representing geometry
- Turbulence generates smaller and smaller scales with larger and larger gradients
- $\partial^2 U_j / \partial x_i^2$ becomes larger until it balances small $1/\mathrm{Re}$
- Dissipation of turbulence by viscous forces at smallest scales

$\Rightarrow$ Energy transfer from large to small scales

Introduction to Turbulence

Statistical Description of Turbulence



Turbulence is random!

<u>Random Process</u>

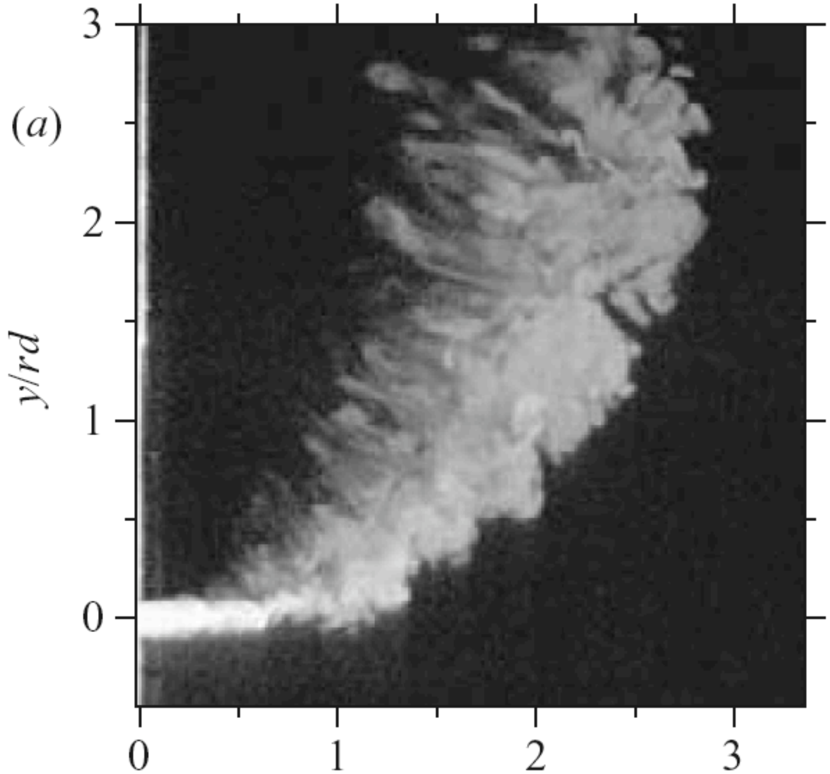
- Process is random if it is non-deterministic
- Random process <u>cannot</u> be predicted

Turbulence theory tries to predict statistical properties or probabilities of turbulence, e. g.:

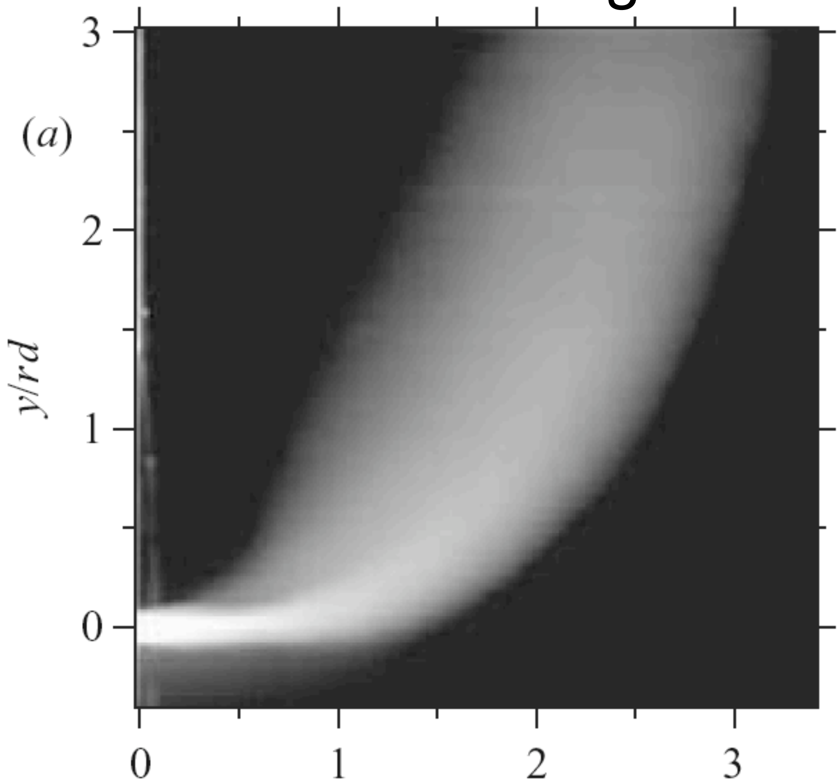
- Mean velocities
- Average velocity fluctuations
- Probability density functions (PDFs)

Example: Time Averaging

Instantaneous



Time average



Statistical Description of Turbulence

Event

- Given random variable U from flow experiment
- Measure statistical behavior of U through certain events
- Event defined as true/false

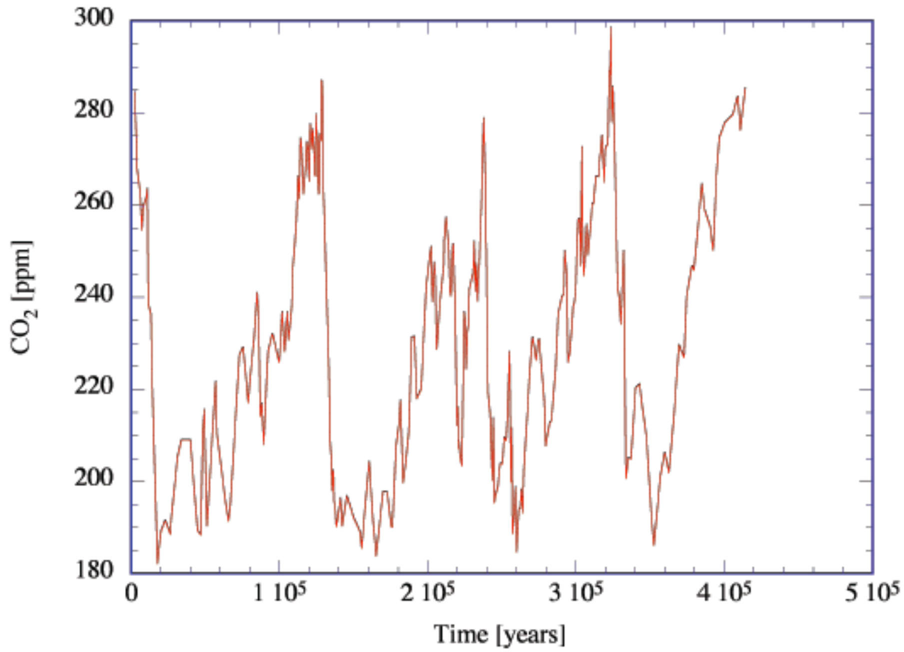
V is called the sample space variable

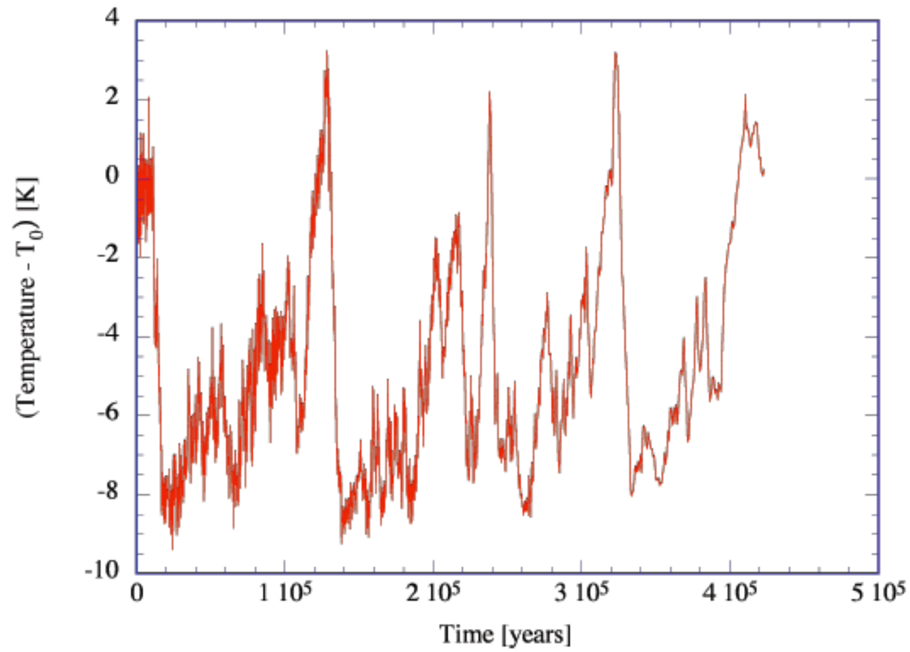
V is used to characterize U

Sample space is a 'scale' to measure random variable

Example:
Random Variables: CO_2 and Temperature

Carbon dioxide mole fraction and temperature as function of time over past 500,000 years





- Behavior of both variables looks quite similar
- Certain time scales can be identified for large and small scale variations
- Magnitudes of fluctuations can be identified for given time scales

Statistical Description of Turbulence

Probability

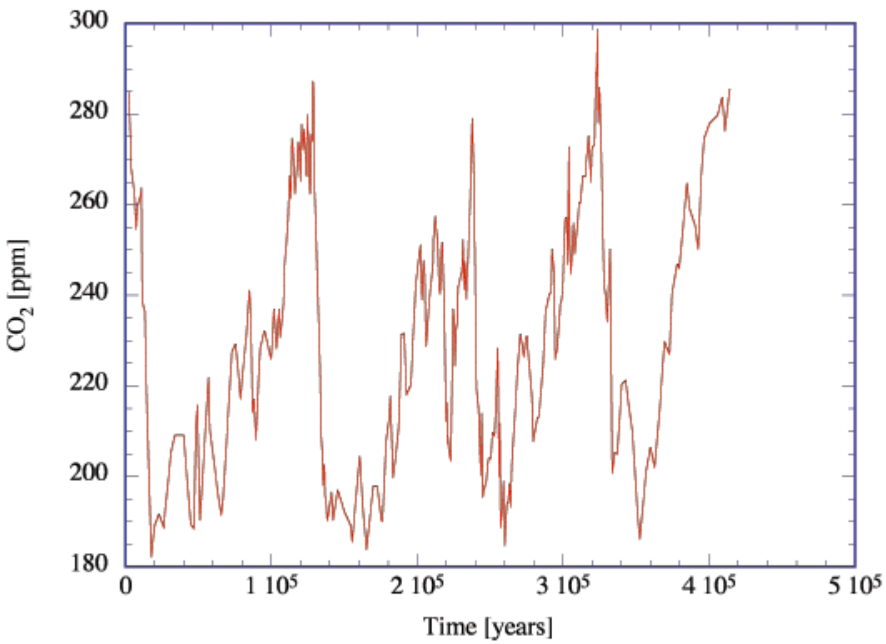
Probability of event E

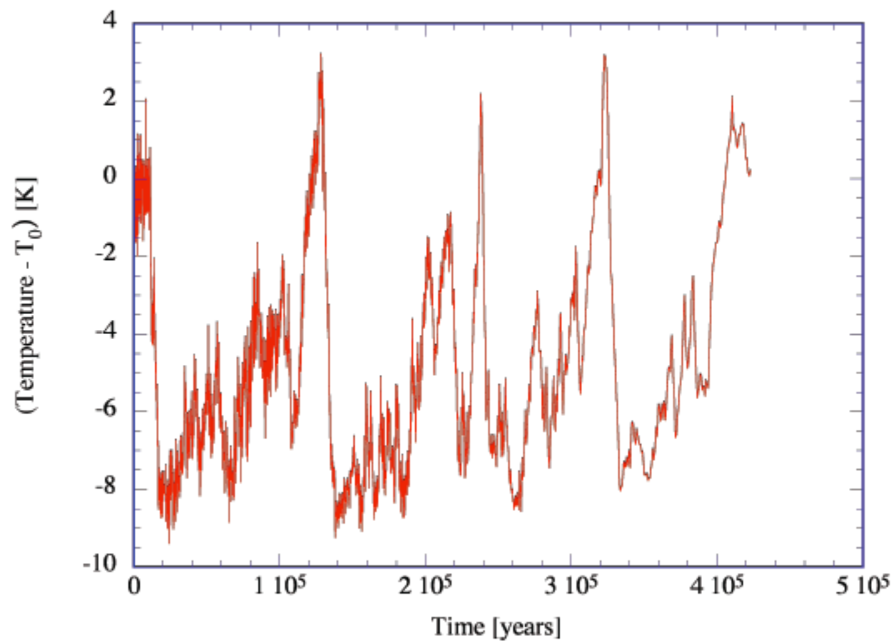
$$P(E) = P\{U < V_b\}$$

$P(E)$ is scalar quantity with $0 \leq P(E) \leq 1$

Example: CO_2 and Temperature

Measure data in events:





- For example:
  - Probability(CO_2 < 200) = 0.15
  - Probability(CO_2 < 220) = 0.4
  - Probability(CO_2 < 260) = 0.75

Statistical Description of Turbulence

Cumulative Distribution Function (CDF)

Definition of CDF

$$F(V) \equiv P\{U < V\}$$

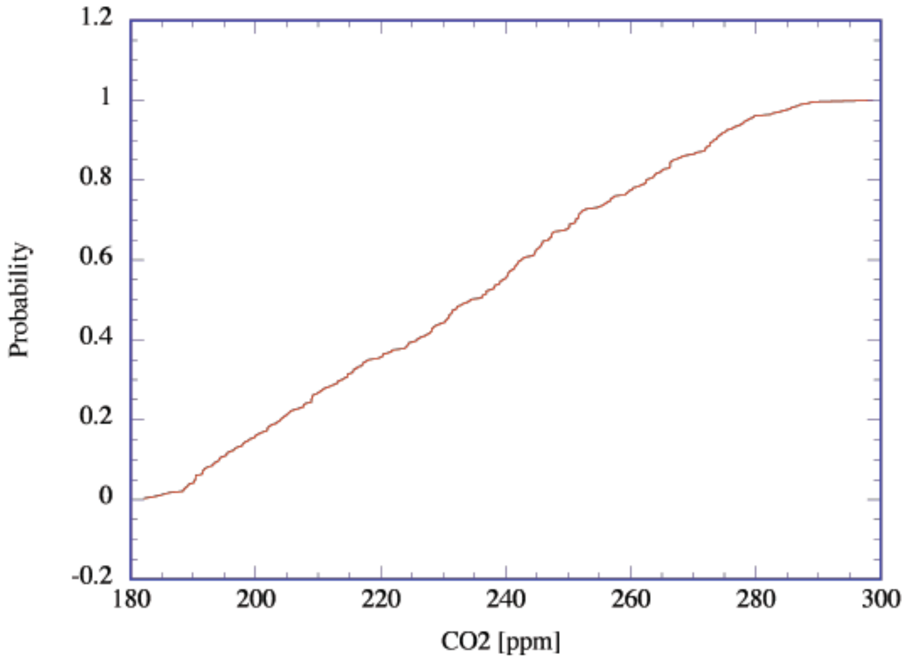
e. g. $\quad P(E) = F(V_b)$

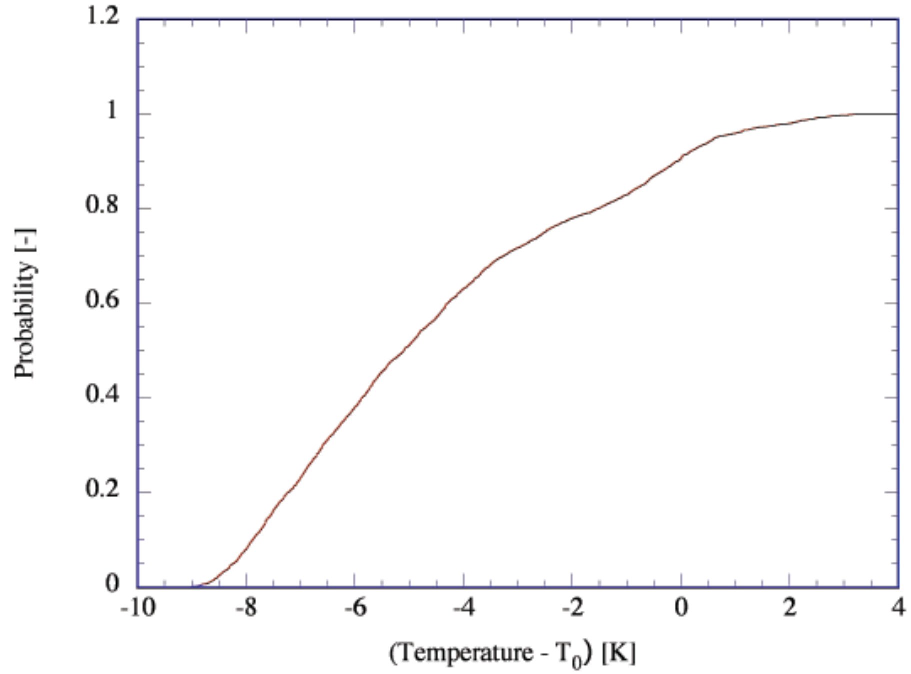
Properties:

$$\begin{aligned} F(-\infty) &= 0 \\ F(\infty) &= 1 \\ V_b > V_a &: \quad F(V_b) \geq F(V_a) \end{aligned}$$

Example: CO_2 and Temperature

Cumulative Distribution Function:





- Looks different for both functions
- Information about where most samples can be expected not too obvious
- Information about correlation of both functions lost

Statistical Description of Turbulence

Probability Density Function (PDF)

Definition of PDF

$$f(V) \equiv \frac{dF(V)}{dV}$$

Properties

$$f(V) \geq 0$$

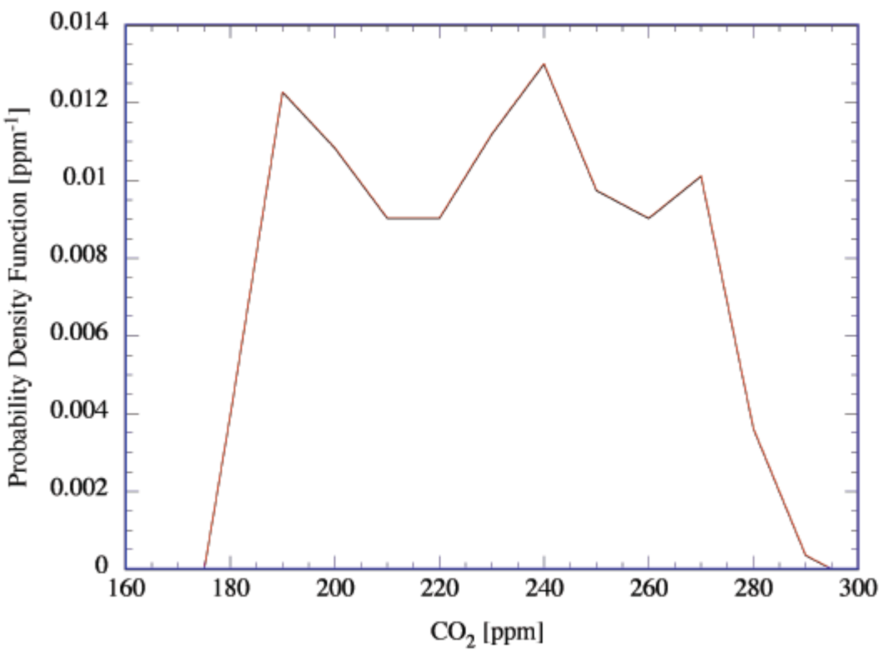
Normalization condition

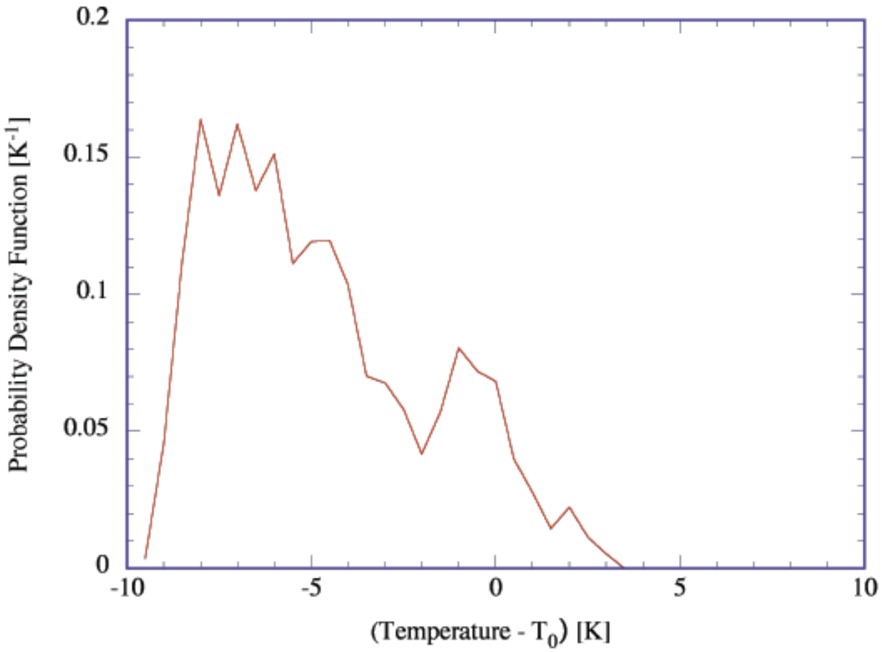
$$\int_{-\infty}^{\infty} f(V)dV = \int_{-\infty}^{\infty} \frac{dF(V)}{dV}dV = [F(V)]_{-\infty}^{\infty} = 1$$

- PDF or CDF fully characterize statistical properties of random variable
- However, U cannot be recovered from PDF
  $\longrightarrow$ no time-scale information

Example: CO_2 and Temperature

Probability Density Function:





- Represents binning
- Large where high probability is expected

Statistical Description of Turbulence

Mean and Moments of PDFs

- Mean of random variable U defined by

$$\langle U \rangle = \int_{-\infty}^{\infty} V f(V) dV$$

- Mean of a function $Q(U)$

$$\langle Q(U) \rangle = \int_{-\infty}^{\infty} Q(V) f(V) dV$$

- Sum of random variables U and W

$$\langle U + W \rangle = \langle U \rangle + \langle W \rangle$$

Statistical Description of Turbulence

- Fluctuation of U

$$u = U - \langle U \rangle$$

- Variance of U

$$\langle u^2 \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^2 f(V) dV$$

- RMS u' or standard deviation σ

$$u' = \sigma = \sqrt{\langle u^2 \rangle}$$

Statistical Description of Turbulence

- Raw moments

$$\langle U^n \rangle = \int_{-\infty}^{\infty} V^n f(V) dV$$

- Central moments

$$\mu_n \equiv \langle u^n \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^n f(V) dV$$

Then

$$\begin{aligned} \mu_0 &= 1 \qquad \text{(normalization condition)} \\ \mu_1 &= 0 \\ \mu_2 &= \sigma^2 \end{aligned}$$

Statistical Description of Turbulence

- Standardization
  - Standardized variable is

$$\hat{U} = \frac{U - \langle U \rangle}{\sigma} \quad \longrightarrow \quad U = \sigma \hat{U} + \langle U \rangle$$

$\hat{U}$ has zero mean and unit variance

  - Standardized PDF

$$\hat{f}(\hat{V}) = \sigma f(\sigma \hat{V} + \langle U \rangle)$$

Statistical Description of Turbulence

– Standardized moments

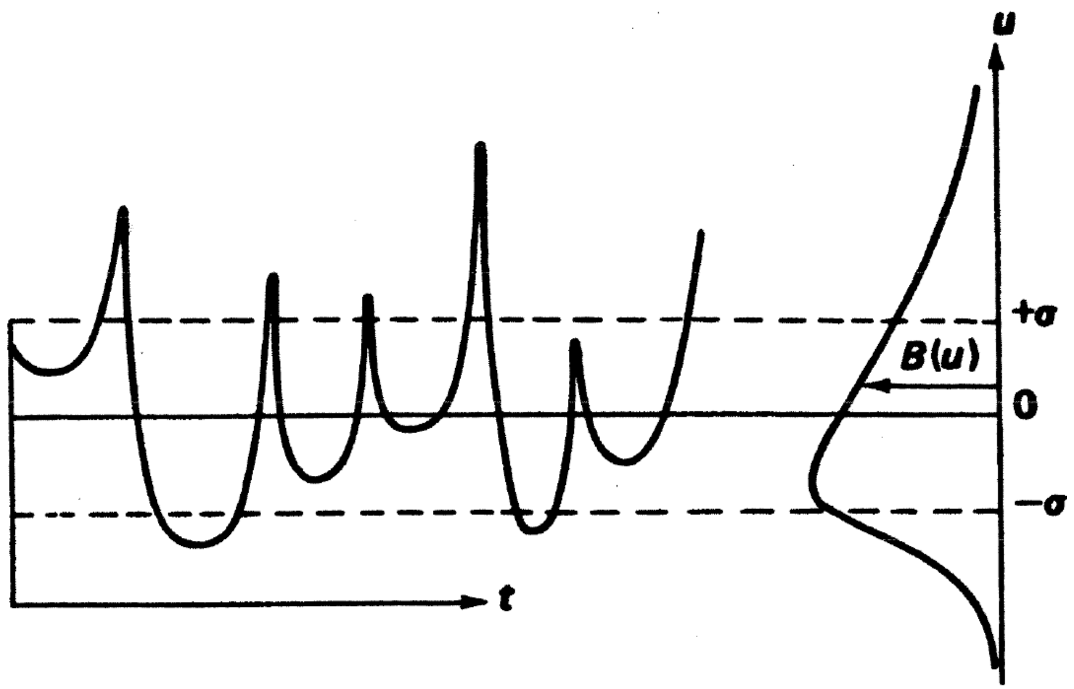
$$\hat{\mu}_n = \int_{-\infty}^{\infty} \hat{V}^n \hat{f}(\hat{V}) d\hat{V}$$

Then

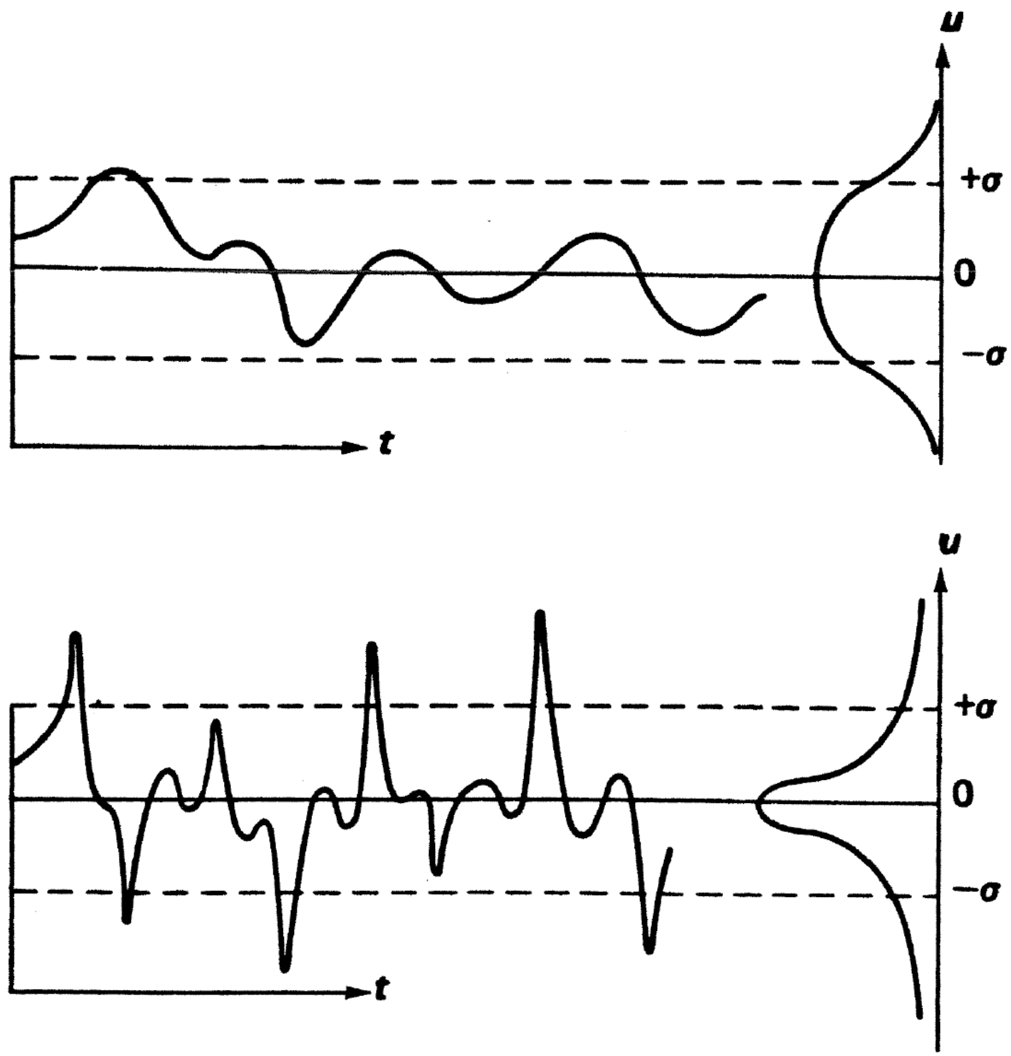
$$\begin{aligned} \hat{\mu}_0 &= 1 \quad \text{(normalization condition)} \\ \hat{\mu}_1 &= 0 \\ \hat{\mu}_2 &= 1 \\ \hat{\mu}_3 &: \quad \text{Skewness} \\ \hat{\mu}_4 &: \quad \text{Flatness or Kurtosis} \end{aligned}$$

Statistical Description of Turbulence

PDF with Positive Skewness



Statistical Description of Turbulence



Kurtosis (Flatness)

Which signal has larger kurtosis?

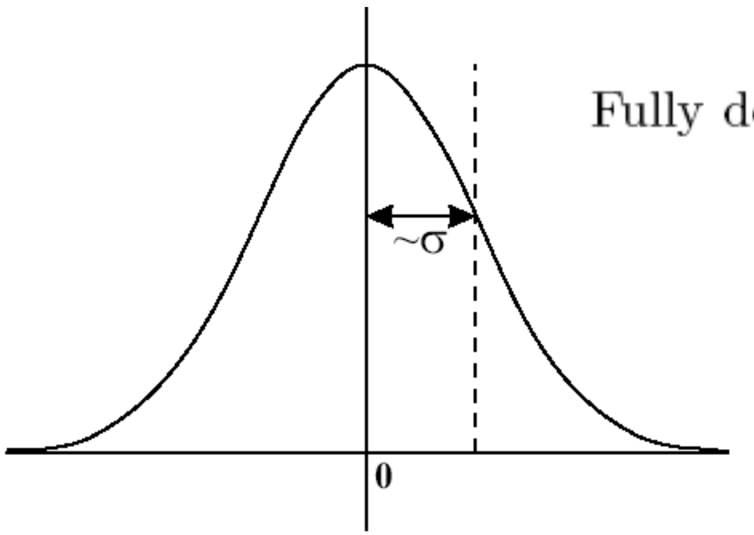
Statistical Description of Turbulence

Examples of Probability Distributions

- Normal Distribution

  Random variable U with standard deviation σ,

$$N(V, \langle U \rangle, \sigma) = f(V) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(V - \langle U \rangle)^2}{2\sigma^2}\right)$$



Fully described by mean and variance:

$$\hat{N}(V) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{V^2}{2}\right)$$

Statistical Description of Turbulence

- Delta Function

$$f(V) = \delta(V - \langle U \rangle) = \begin{cases} \infty & \text{if } V = \langle U \rangle \\ 0 & \text{otherwise} \end{cases}$$

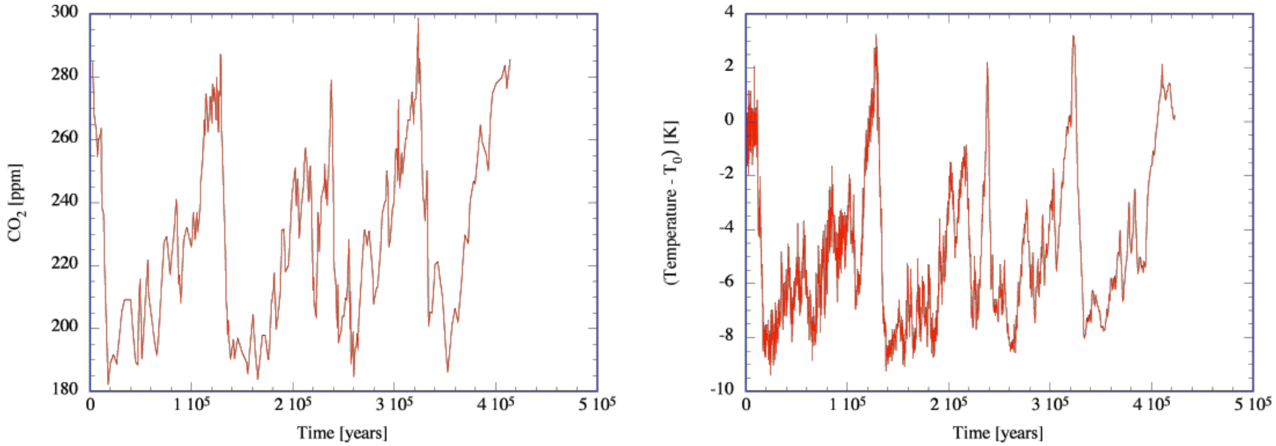
and

$$\int \delta(V - \langle U \rangle) = 1$$

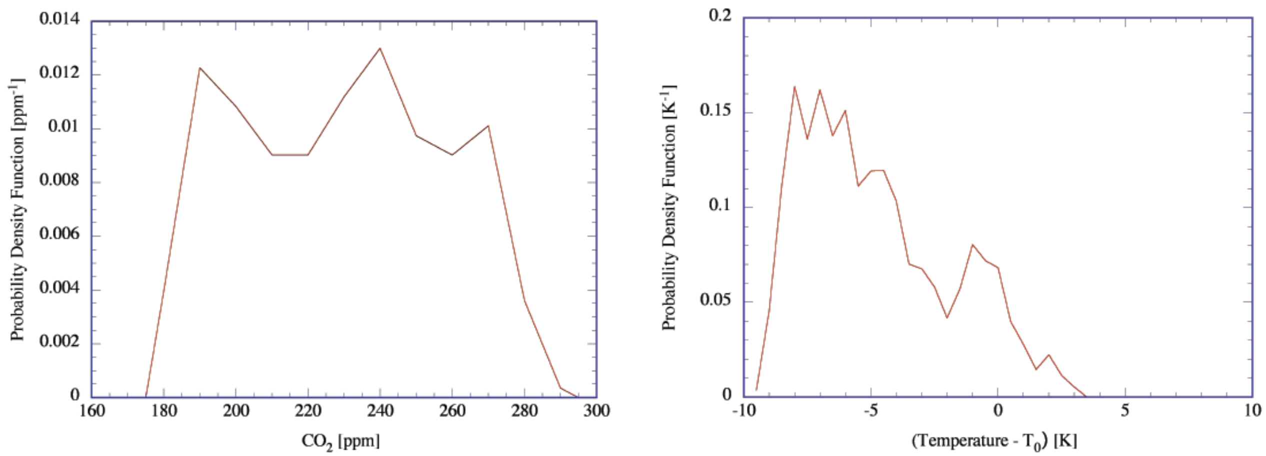
$$\delta(V - \langle U \rangle) = \lim_{\sigma \to 0} N(V, \langle U \rangle, \sigma)$$

Example: CO_2 and Temperature

CO_2 and temperature time history



CO_2 and temperature PDFs



Statistical Description of Turbulence

Joint Random Variables

Given two random variables U_1, U_2 with sample space variables V_1 and V_2

Joint CDF

Defined as

$$F_{12}(V_1, V_2) = P\{U_1 < V_1, U_2 < V_2\}$$

Properties:

- Non-decreasing function
- $F_{12}(-\infty, V_2) = F_{12}(V_1, -\infty) = 0$
- $F_{12}(\infty, V_2) = F(V_2)$ marginal (CDF)

Statistical Description of Turbulence

Joint PDF

Defined as:

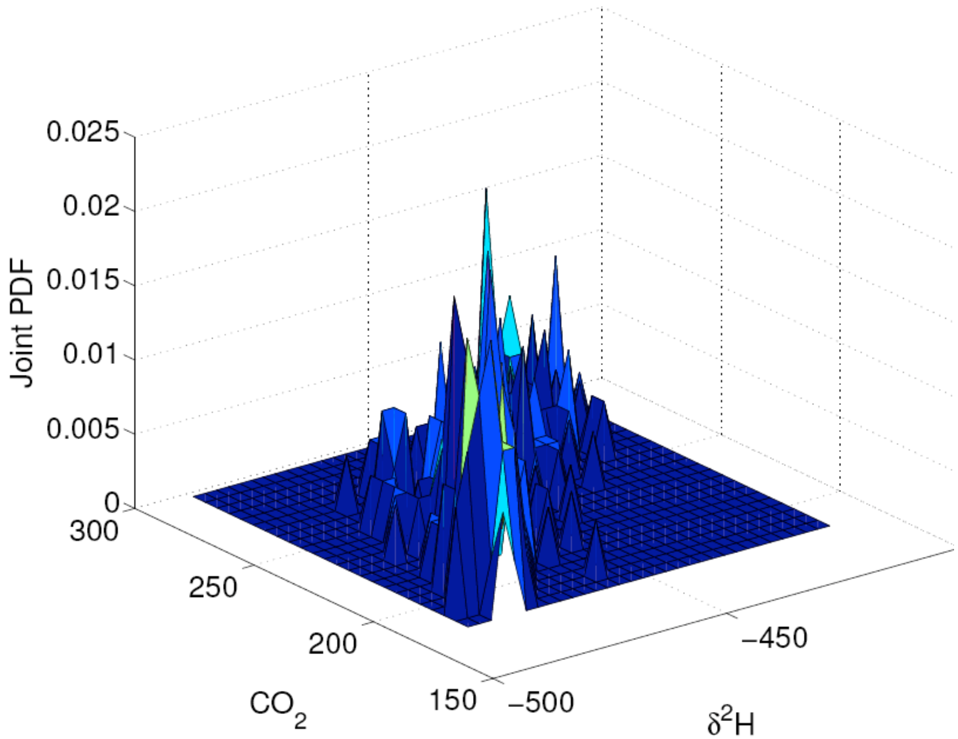
$$f_{12}(V_1, V_2) = \frac{\partial^2 F_{12}(V_1, V_2)}{\partial V_1 \partial V_2}$$

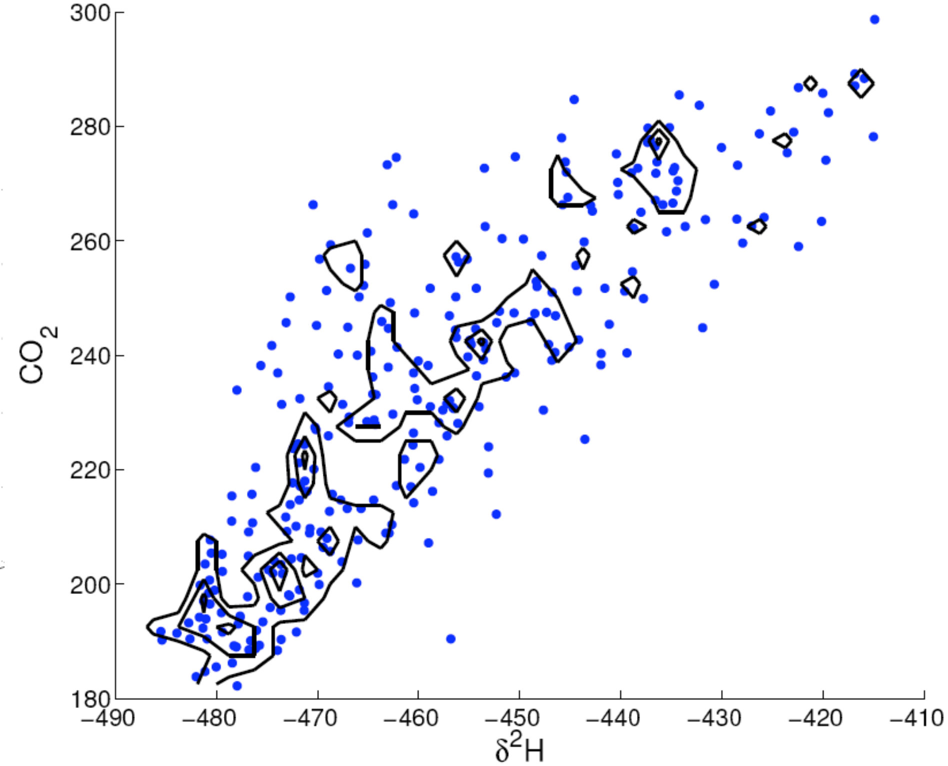
Properties:

- $f_{12} \geq 0$
- $\int_{-\infty}^{\infty} f_{12}(V_1, V_2) dV_1 = f_2(V_2)$ (marginal PDF)
- $\iint_{-\infty}^{\infty} f_{12}(V_1, V_2)\, dV_1 dV_2 = 1$

Example: CO_2 and Temperature

Joint PDF of CO_2 and Temperature





- Probability of high temperature increases with probability for high CO_2
- Temperature increase seems to be stronger at higher values of CO_2

Statistical Description of Turbulence

For any function $Q(U_1, U_2)$

$$\langle Q(U_1, U_2) \rangle = \iint_{-\infty}^{\infty} Q(V_1, V_2)\, f_{12}(V_1, V_2)\, dV_1 dV_2$$

From this follows

$$\langle U_1 \rangle \,, \quad \langle U_2 \rangle \,, \quad \langle U_1^2 \rangle \,, \quad \underbrace{\langle u_1 u_2 \rangle}_{\text{co-variance}} \quad , \ldots$$

Statistical Description of Turbulence

Correlation Coefficient

$$\rho_{12} \equiv \frac{\langle u_1 u_2 \rangle}{\left(\langle u_1^2 \rangle \langle u_2^2 \rangle \right)^{\frac{1}{2}}}$$

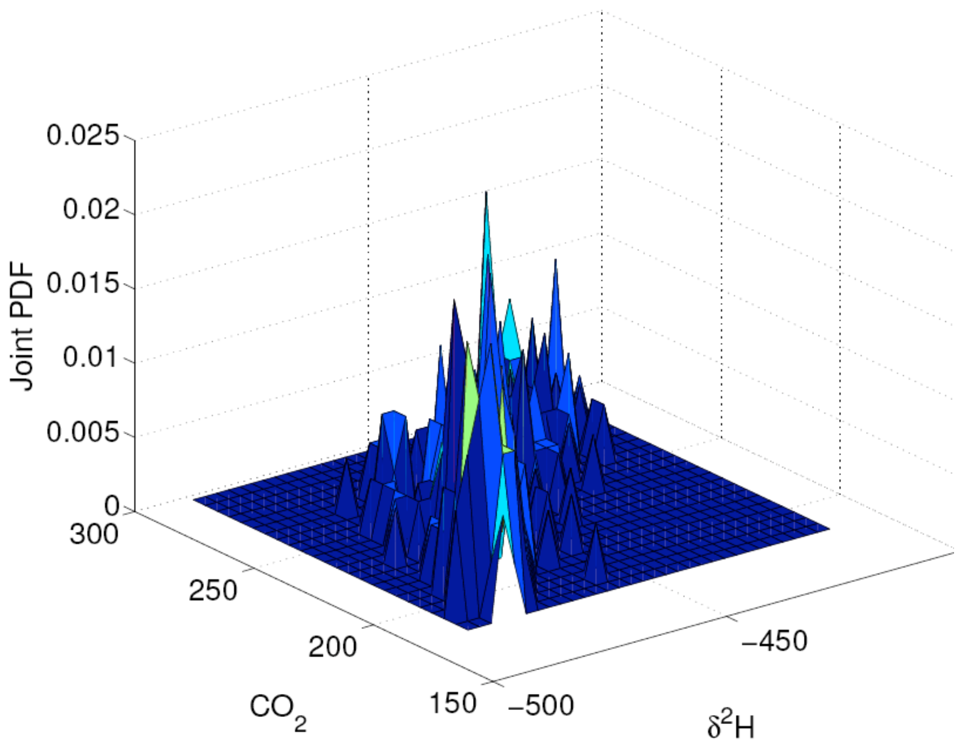
Properties:

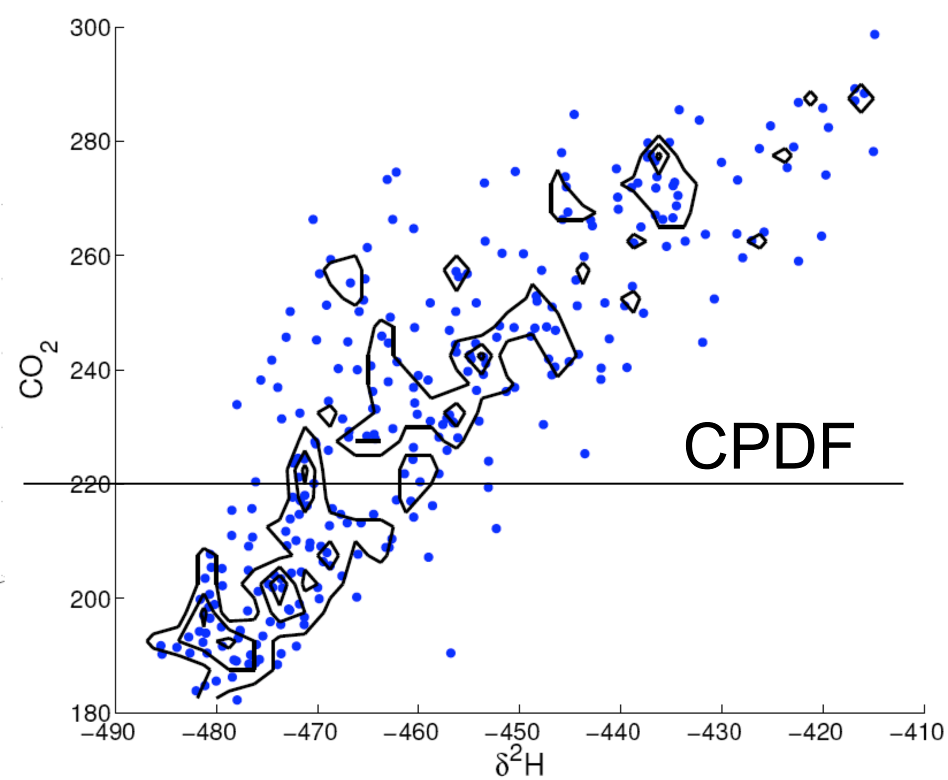
- $-1 \leq \rho_{12} \leq 1$
- $\rho_{12} = 0 \Rightarrow u_1, u_2$ uncorrelated
- $\rho_{12} = \pm 1 \Rightarrow u_1, u_2$ perfectly correlated

(Correlation coefficient for CO_2/temperature example = 0.839)

Example: CO_2 and Temperature

Conditional PDF of Temperature





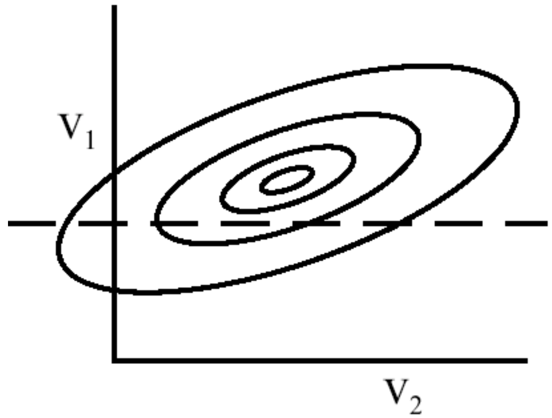
Statistical Description of Turbulence

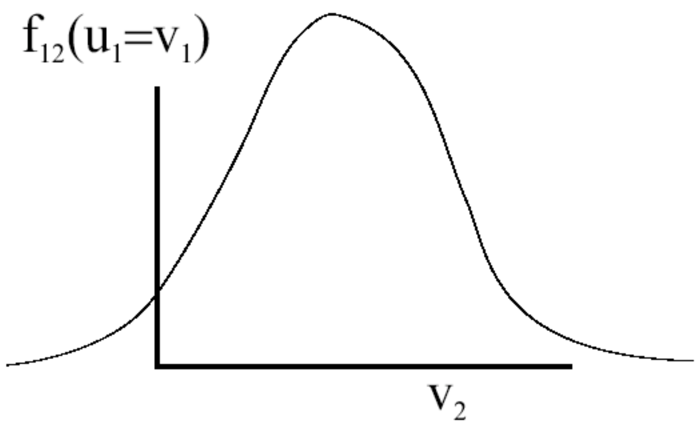
Conditional PDF

$$f_{2|1}(V_2|U_1 = V_1) = f_{2|1}(V_2|V_1) = \frac{f_{12}(V_1, V_2)}{f(V_1)} \quad \text{(Bayes's Theorem)}$$

$$\int f_{12}(V_1, V_2) dV_2 = f_1(V_1)$$

$$\Rightarrow \int f_{2|1}(V_2|V_1)\, dV_2 = \int \frac{f_{12}(V_1, V_2)}{f(V_1)} dV_2 = \frac{f(V_1)}{f(V_1)} = 1$$





Statistical Description of Turbulence

For U_1 and U_2 independent

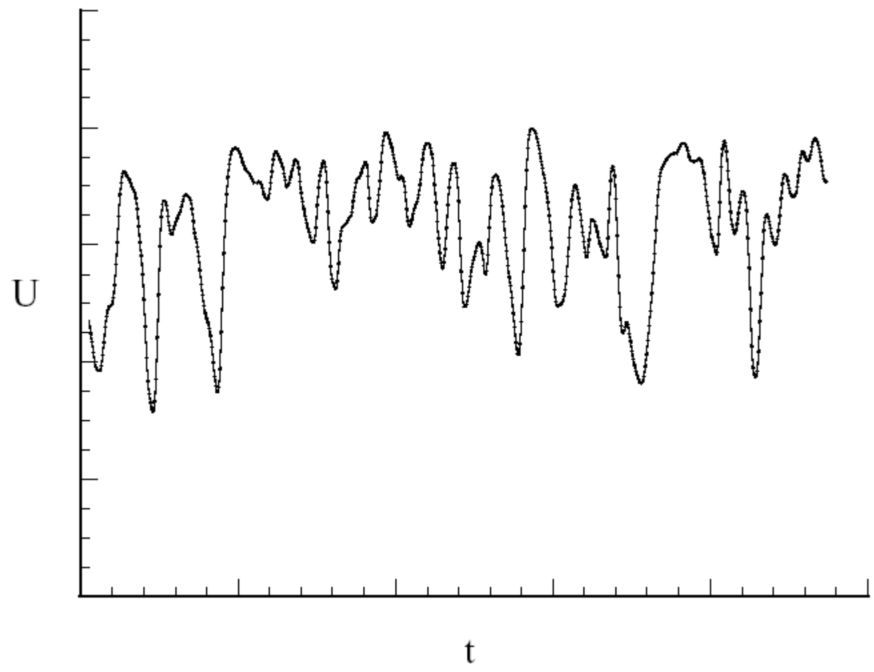
$$f_{12}(V_1, V_2) = f(V_2|V_1) f(V_1) = f(V_2)\, f(V_1)$$

<u>Conditional average</u>

$$\langle Q(U_1, U_2) | U_1 = V_1 \rangle = \langle Q(U_1, U_2) | V_1 \rangle = \int Q(V_1, V_2)\, f_{2|1}(V_2|V_1) dV_2$$

Random Processes, Random Fields, and Two-Point Correlations

- Consider temporal development of statistically stationary random process given as $U(t)$



- Marginal PDF of $U(t)$ does not include time-scale
  $\longrightarrow$ use joint PDF of $U(t)$ at multiple times

Random Processes, Random Fields, and Two-Point Correlations

Two-Time Joint CDF

$$F(V_1, t_1, V_2, t_2) = P\{U(t_1) < V_1, U(t_2 < V_2)\}$$

Two-Time Joint PDF

$$f(V_1, t_1, V_2, t_2) = \frac{\partial^2 F}{\partial V_1 \partial V_2}$$

For statistically stationary flows

$$f(V_1, t_1, V_2, t_2) = f(V_1, V_2, \Delta t)$$

Random Processes, Random Fields, and Two-Point Correlations

Definition of Simple Statistical Measures for Random Processes

- Auto covariance

$$R(s) \equiv \langle u(t)u(t+s)\rangle$$

- Autocorrelation function

$$\rho(s) = \frac{\langle u(t)u(t+s)\rangle}{\langle u^2(t)\rangle}$$

Properties:

$$\begin{aligned} |\rho(s)| &\leq 1 \\ \rho(0) &= 1 \\ \rho(s) &= \rho(-s) \end{aligned}$$

Random Processes, Random Fields, and Two-Point Correlations

- Integral time-scale

$$\overline{\tau} = \int_0^\infty \rho(s) ds$$

- Frequency spectrum is Fourier transform of (twice) the autocovariance

$$E(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} R(s) e^{-i\omega s} ds = \frac{2}{\pi} \int_0^\infty R(s) e^{-i\omega s} ds$$

Contribution to variance from frequency range $\omega_a \leq \omega \leq \omega_b$ given by

$$\int_{\omega_a}^{\omega_b} E(\omega) d\omega$$

Random Processes, Random Fields, and Two-Point Correlations

Random Fields

Consider random vector field $\boldsymbol{U}(\boldsymbol{x},t)$

One-point, one-time JCDF

$$F(\boldsymbol{V},\boldsymbol{x},t) = P\{U_i(\boldsymbol{x},t) < V_i, \;\; i=1,2,3\}$$

One-point, one-time JPDF

$$f(\boldsymbol{V};\boldsymbol{x},t) = \frac{\partial^3 F}{\partial V_1 \partial V_2 \partial V_3}$$

Random Processes, Random Fields, and Two-Point Correlations

Mean U_1-velocity

$$\langle U_1(\boldsymbol{x},t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_1 f(\boldsymbol{V};\boldsymbol{x},t)dV_1dV_2dV_3$$

Mean $\boldsymbol{U}$-velocity tensor

$$\langle \boldsymbol{U}(\boldsymbol{x},t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{V} f(\boldsymbol{V};\boldsymbol{x},t)dV_1dV_2dV_3$$

Velocity fluctuation

$$\boldsymbol{u}(\boldsymbol{x},t) \equiv \boldsymbol{U}(\boldsymbol{x},t) - \langle \boldsymbol{U}(\boldsymbol{x},t)\rangle$$

Random Processes, Random Fields, and Two-Point Correlations

Two-Point Correlations

Two-point correlations of random velocity field

$$R_{ij}(\boldsymbol{r}, \boldsymbol{x}, t) \equiv \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x} + \boldsymbol{r}, t) \rangle$$

Integral length scales, e.g.

$$L_{11} = \frac{1}{R_{11}(0, \boldsymbol{x}, t)} \int_0^\infty R_{11}(\boldsymbol{e_1} r, \boldsymbol{x}, t) dr$$

Random Processes, Random Fields, and Two-Point Correlations

Velocity spectrum tensor

- For homogeneous turbulence, Fourier transform of $R_{ij}(\boldsymbol{r}, t)$ is velocity spectrum tensor

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ij}(\boldsymbol{r}, t) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} dr_1 dr_2 dr_3$$

$\boldsymbol{\kappa}$ is wave number vector with wave length $\lambda = 2\pi/|\boldsymbol{\kappa}|$

- Inverse transform

$$R_{ij}(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\kappa_1 d\kappa_2 d\kappa_3$$

Random Processes, Random Fields, and Two-Point Correlations

Energy spectrum function

$$E(\kappa, t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ii}(\boldsymbol{\kappa}, t) \delta(|\boldsymbol{\kappa}| - \kappa) d\kappa_1 d\kappa_2 d\kappa_3$$

Turbulent kinetic energy

$$k = \frac{1}{2} \langle u_i^2 \rangle = \int_0^{\infty} E(\kappa) d\kappa$$

Statistical Description of Turbulence: Simplifications

Statistical stationarity

- All statistics are invariant under a shift in time

$$f(\ldots;\quad \boldsymbol{x},\quad t) = f(\ldots;\quad \boldsymbol{x},\quad t + \Delta t)$$

Statistically homogeneous turbulence

- All statistics of velocity fluctuations are invariant under translation of the coordinate system.

$$f(\boldsymbol{u}_i, \text{multi-point}, \text{multi-time}; \boldsymbol{x}, t) = f(\boldsymbol{u}_i, \text{multi-point}, \text{multi-time}; \boldsymbol{x} + \Delta \boldsymbol{x}, t)$$

for any $\Delta \boldsymbol{x}$. This gives

$$\frac{\partial \langle U_i \rangle}{\partial x_j} = \text{const}$$

Statistical Description of Turbulence: Simplifications

Isotropic turbulence

- All statistics are invariant under translation, rotation, and reflection of the coordinate system.

Mean Flow Equations

Reynolds decomposition

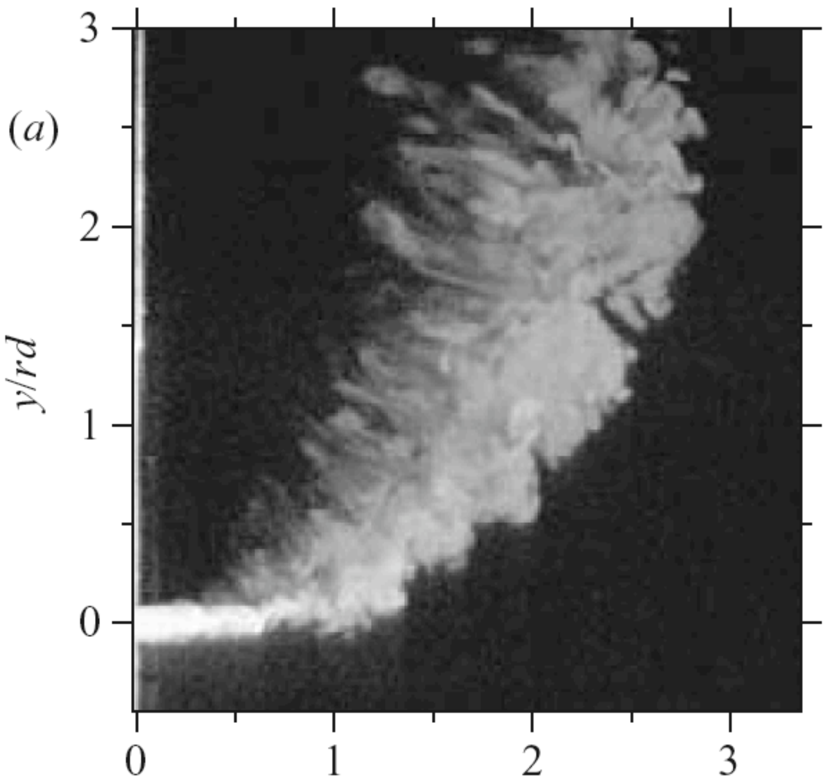
$$\boldsymbol{U}(\boldsymbol{x},t) = \langle \boldsymbol{U}(\boldsymbol{x},t) \rangle + \boldsymbol{u}(\boldsymbol{x},t)$$

- Fluctuation is random quantity
- Desired (not necessary) properties of mean:
  - Deterministic quantity
  - Commutes with differentiation

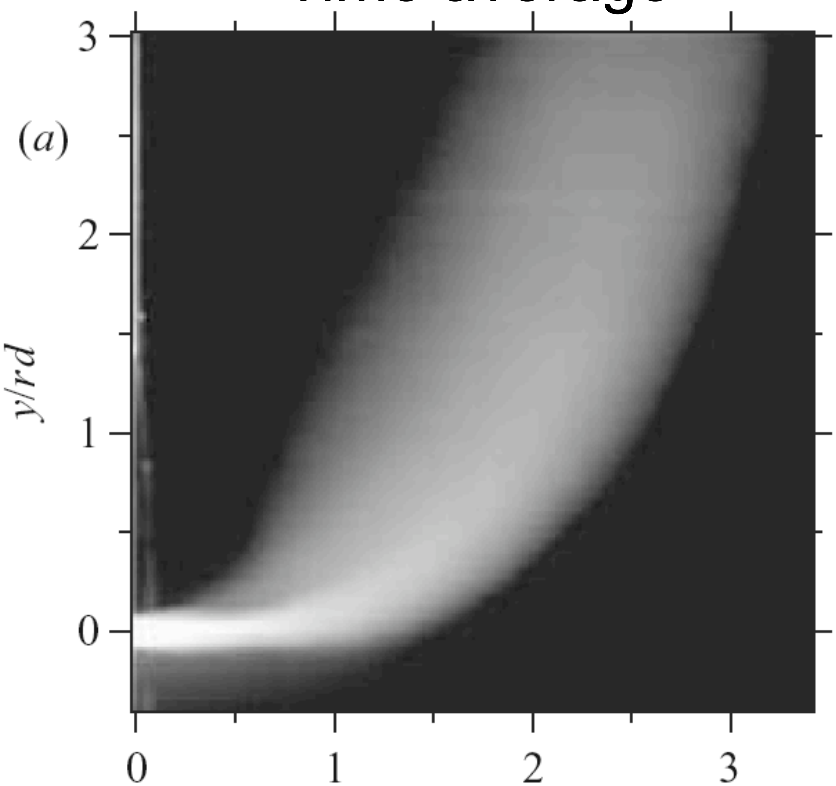
Mean Flow Equations

Time or Ensemble Averaging

Instantaneous



Time average



Mean Flow Equations

Definition of Mean:

1. Mean from pdf

$$\langle U(\boldsymbol{x},t)\rangle \equiv \int_{-\infty}^{\infty} V f(V;\boldsymbol{x},t)dV$$

Mean Flow Equations

All three have these properties:

1. Commute with differentiation
2. $\langle\langle\phi\rangle\rangle = \langle\phi\rangle$
3. $\langle u\rangle = 0$
4. $\langle a + b\rangle = \langle a\rangle + \langle b\rangle$

Mean Flow Equations

Reynolds stress tensor $\langle \boldsymbol{uu} \rangle$ or $\langle u_i u_j \rangle$

Rewriting momentum equation

$$\rho \frac{\partial \langle U_j \rangle}{\partial t} + \rho \langle U_i \rangle \cdot \frac{\partial \langle U_j \rangle}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \langle P \rangle \delta_{ij} - \rho \langle u_i u_j \rangle \right]$$

- $\langle u_i u_j \rangle$ has similar forms as viscous stress tensor
- Momentum transfer by fluctuating velocity
- Reynolds stresses unclosed

⇨ Closure problem!

Mean Flow Equations

Derived quantities:

- Turbulent kinetic energy

$$k = \frac{1}{2} \langle u_i^2 \rangle$$

- Anisotropic tensor

$$\alpha_{ij} = \langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij} \quad \Rightarrow \quad \alpha_{ii} = 0$$

Only anisotropic part of RST is effective in turbulent transport

$$\rho \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{\partial \langle P \rangle}{\partial x_j} = \rho \frac{\partial \alpha_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j} \underbrace{\left(\langle P \rangle + \frac{2}{3} k \right)}_{p*}$$

where $\langle P \rangle + \frac{2}{3} k$ is p^*

Mean Flow Equations

Gradient transport and eddy viscocity

Boussinesq approximation:
(analogous to relations between viscous stress and rate of strain)

$$-\rho\alpha_{ij} = -\rho\langle u_i u_j\rangle + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_t\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right) = 2\rho\nu_t\overline{S_{ij}}$$

where ν_t is eddy viscosity and $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$ is strain rate tensor.

Mean Flow Equations

Mean modeled momentum equation

$$\frac{\partial \langle U_j \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} = -\frac{1}{\rho} \frac{\partial}{\partial x_j}(\langle P \rangle + \frac{2}{3}k) + \frac{\partial}{\partial x_i}\left[\nu_{\text{eff}}\left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i}\right)\right]$$

with $\nu_{\text{eff}} = \nu + \nu_t$

Done!

Except, we don't know ν_t

Mean Flow Equations

Prandtl's Mixing Length Model

Model for eddy viscosity

Assumption: Turbulent transport due to eddy with length scale l_m and velocity fluctuation u^*

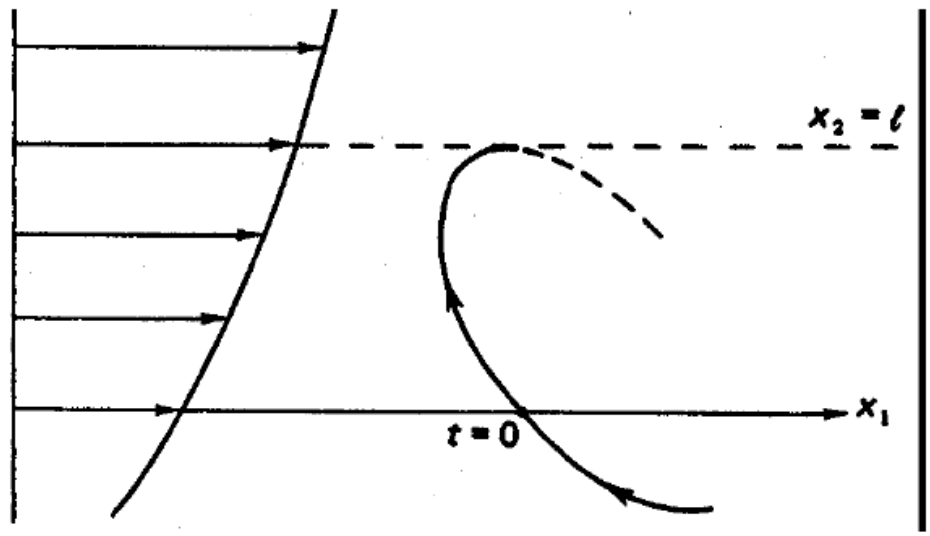
Then by dimensional arguments

$$\nu_t \sim l^* u^*,$$

$$l^* = l_m$$

Mean Flow Equations

The velocity fluctuation is approximated from the following simplified view



For example:

The turbulent eddy will transport fluid from a region 1 with low mean momentum $\rho \langle U \rangle_1$ over a distance l_m to a region 2 with larger mean momentum $\rho \langle U \rangle_2$ creating fluctuations in momentum ρu^*

Mean Flow Equations

The momentum fluctuation ρu^* can be approximated by the difference in mean momentum as

$$\rho u^* = \rho \langle U \rangle (x_2) - \rho \langle U \rangle (x_1)$$

Tayler series for $\langle U \rangle (x_2)$

$$\langle U \rangle (x_2) = \langle U \rangle (x_1) + \frac{\partial \langle U \rangle}{\partial x} l_m$$

gives

$$u^* = \left| \frac{\partial \langle U \rangle}{\partial x} \right| l_m$$

and

$$\nu_t = l_m^2 \left| \frac{\partial \langle U \rangle}{\partial x} \right|$$

l_m needs to be specified

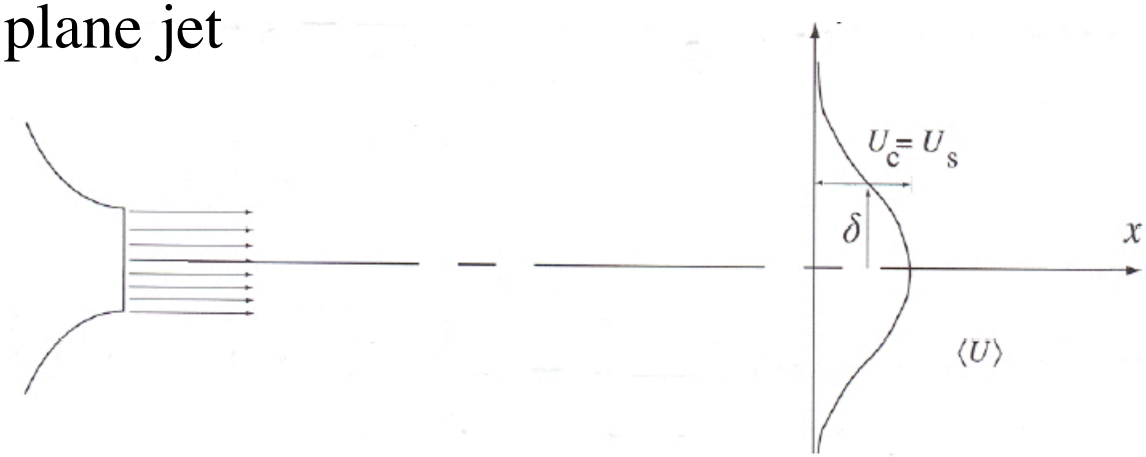
Free Shear Flows

Why free shear flows and why round jet?

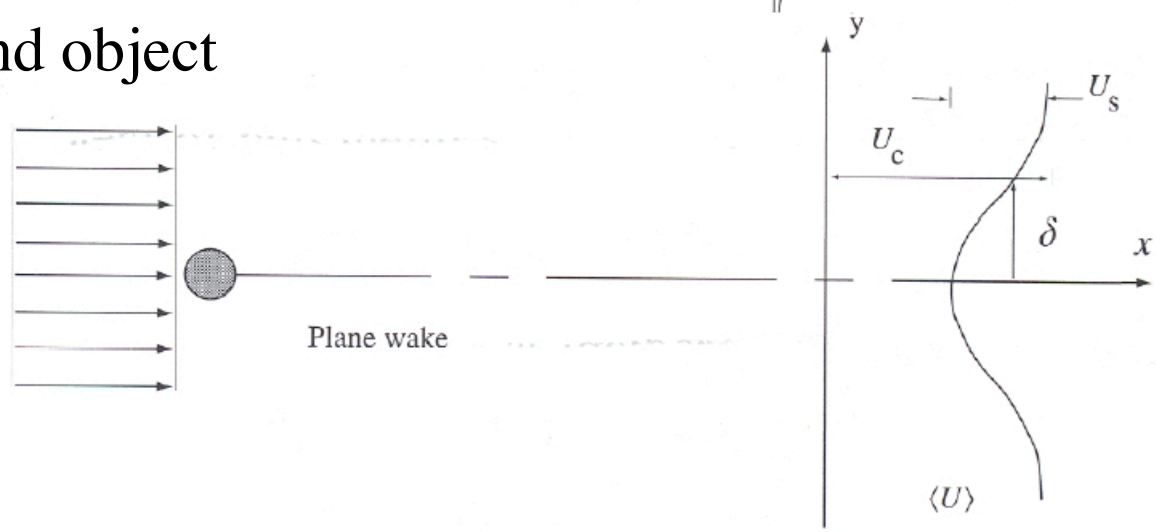
- Distinguish between wall-bounded flows and non-wall bounded flows
- Start with free shear flow
- Round jet is one canonical example of free shear flow
- Round jet convenient because of self-similar solution
- Use this example to look at
    - Scaling
    - Reynolds stresses
    - Turbulent kinetic energy
    - Turbulent kinetic energy equation
    - Turbulent kinetic energy budget

Free Shear Flows: Examples

Round or plane jet

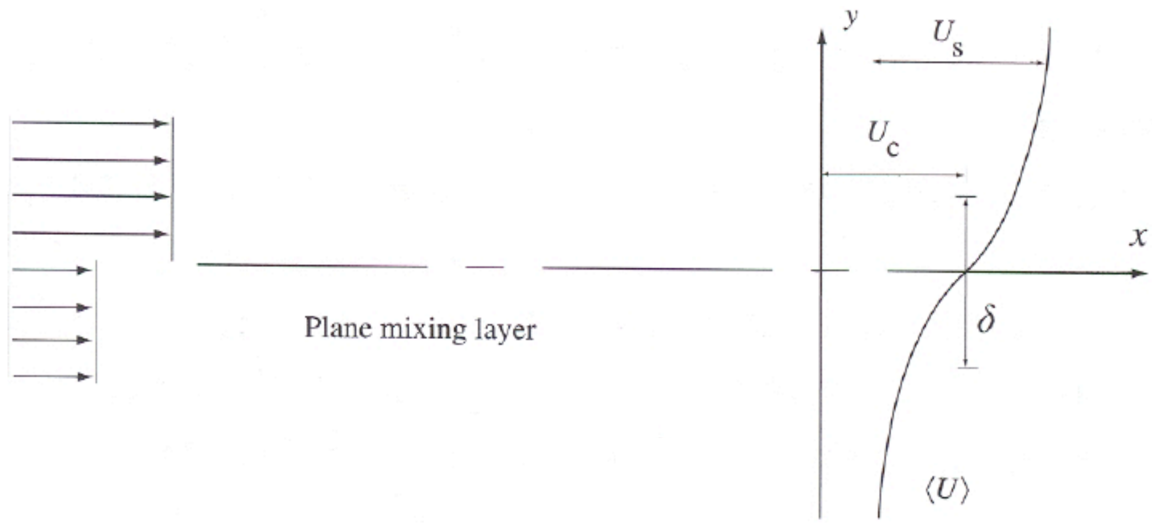


Wake behind object

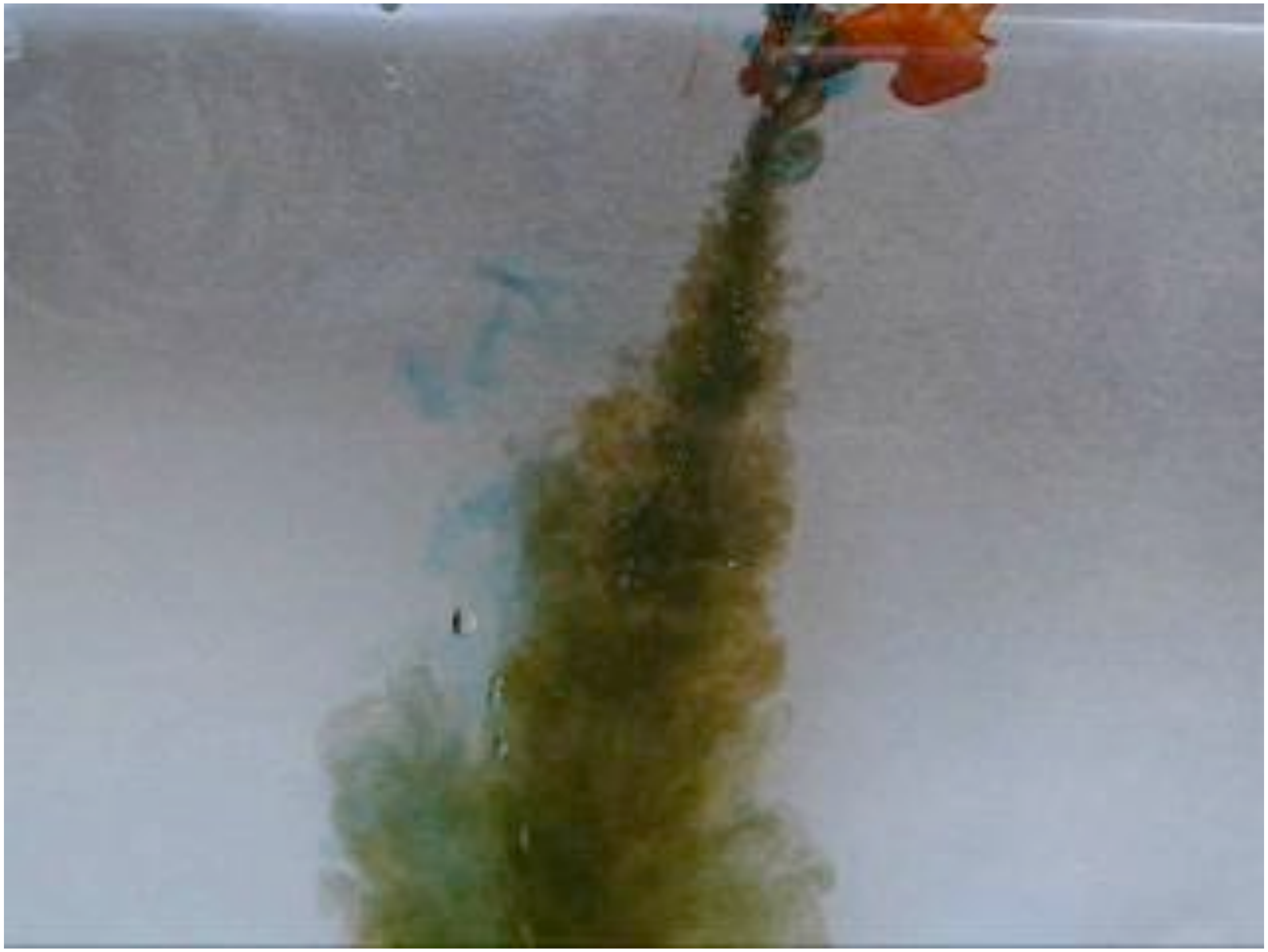


Free Shear Flows: Examples

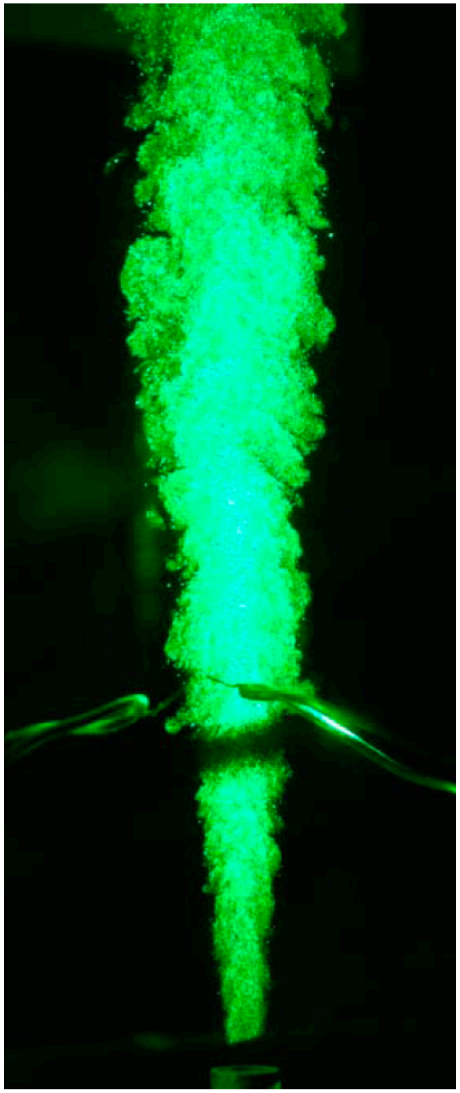
Temporal or spatial mixing layer



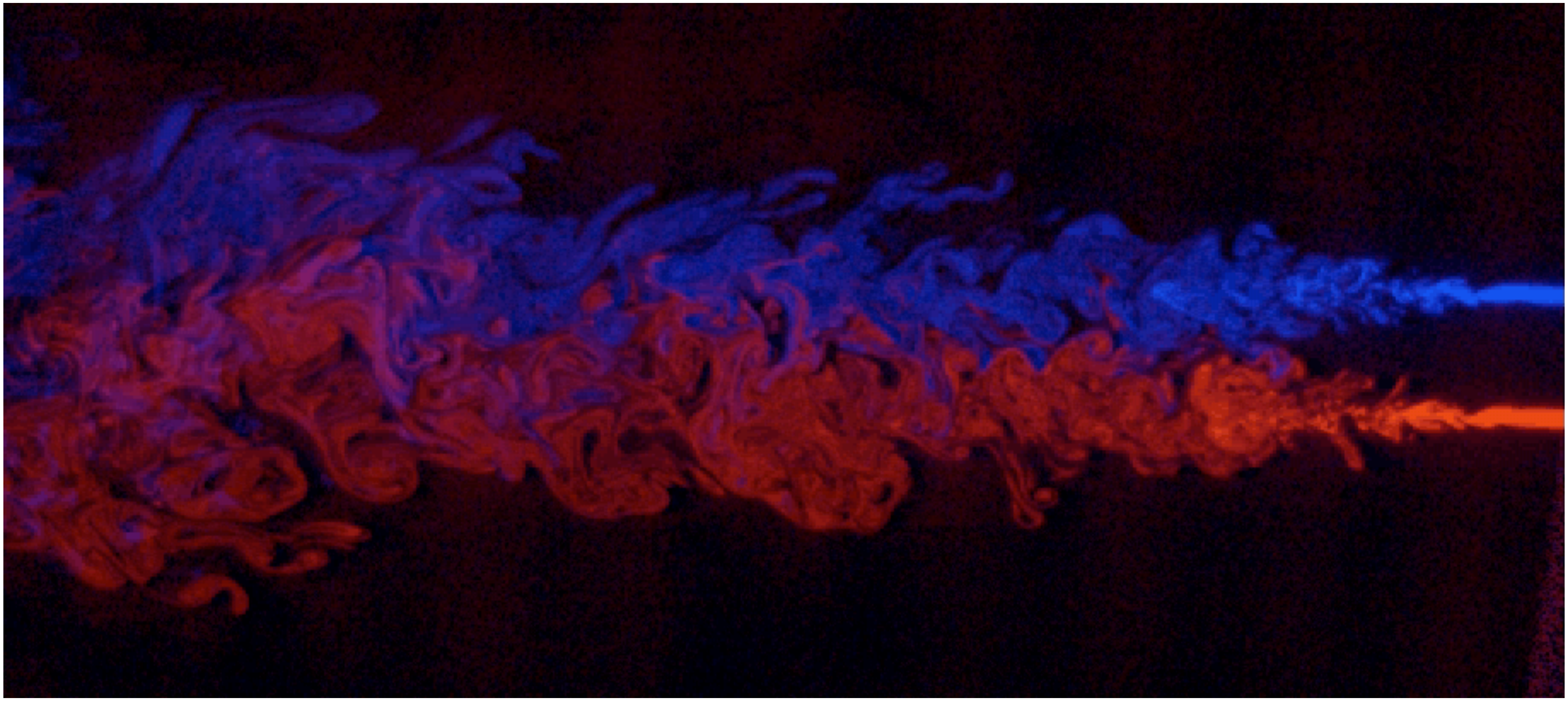
Free Shear Flows: Round Jet



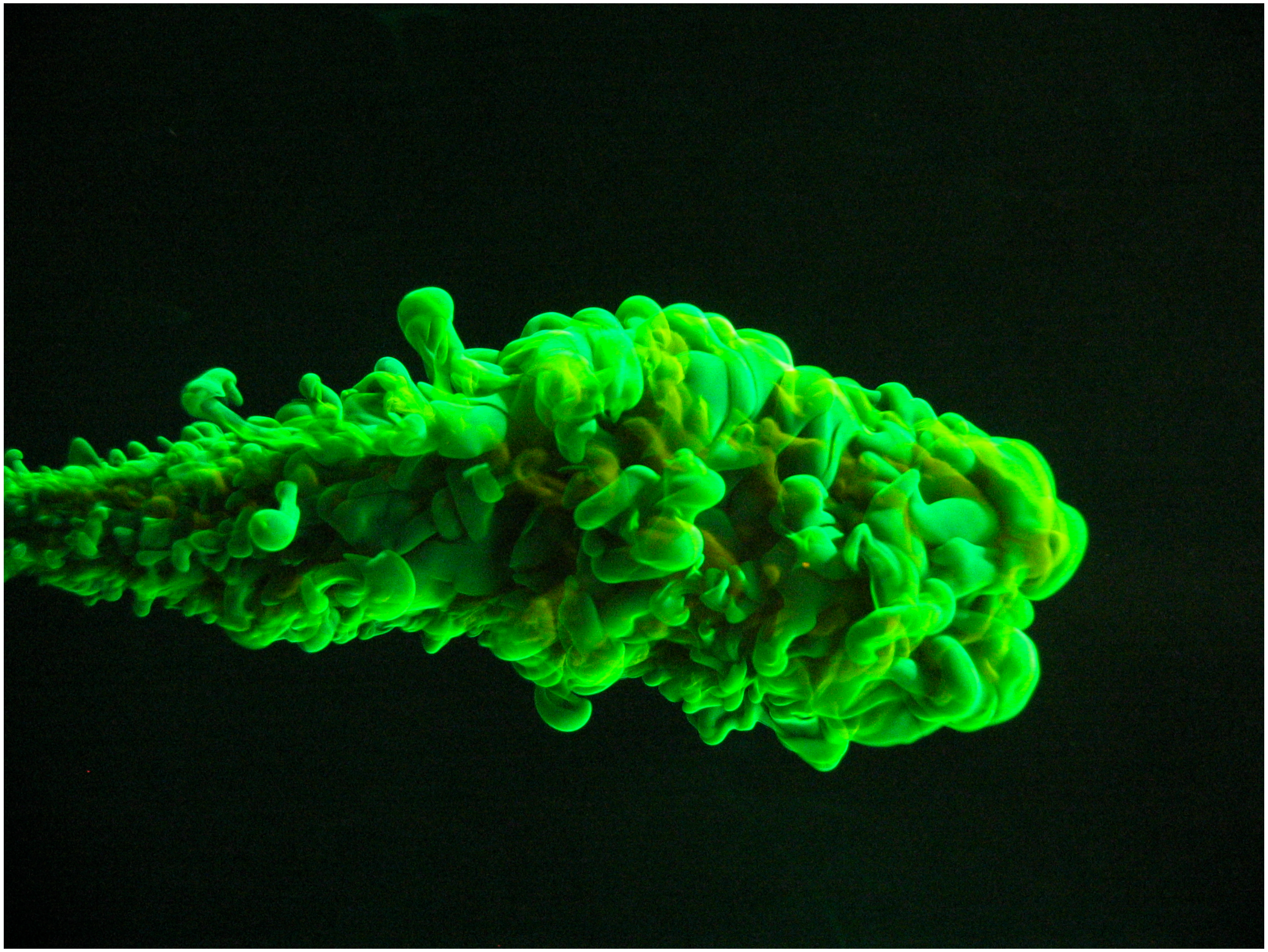
Free Shear Flows: Round Jet



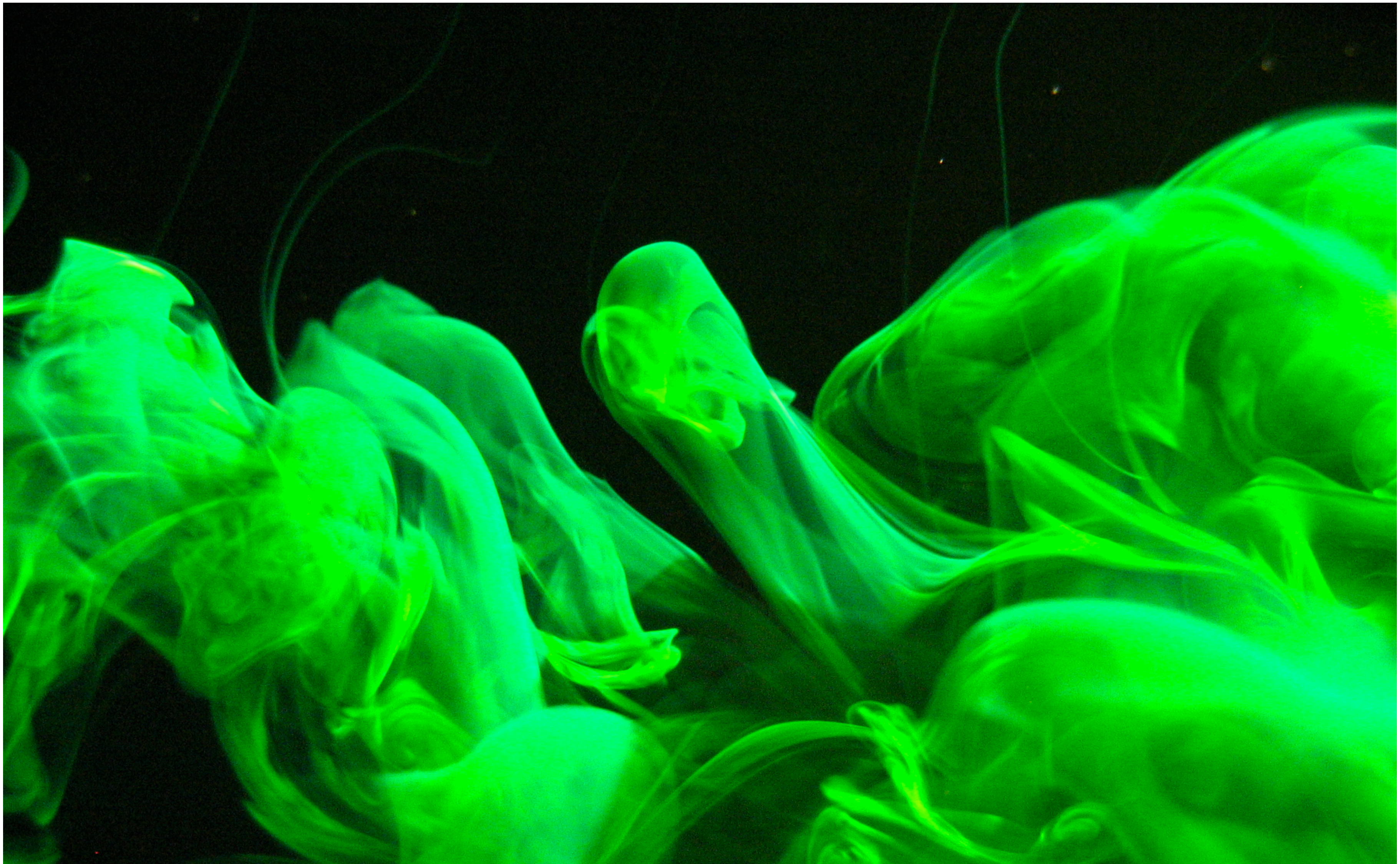
Free Shear Flows: Round Jet



Free Shear Flows: Round Jet

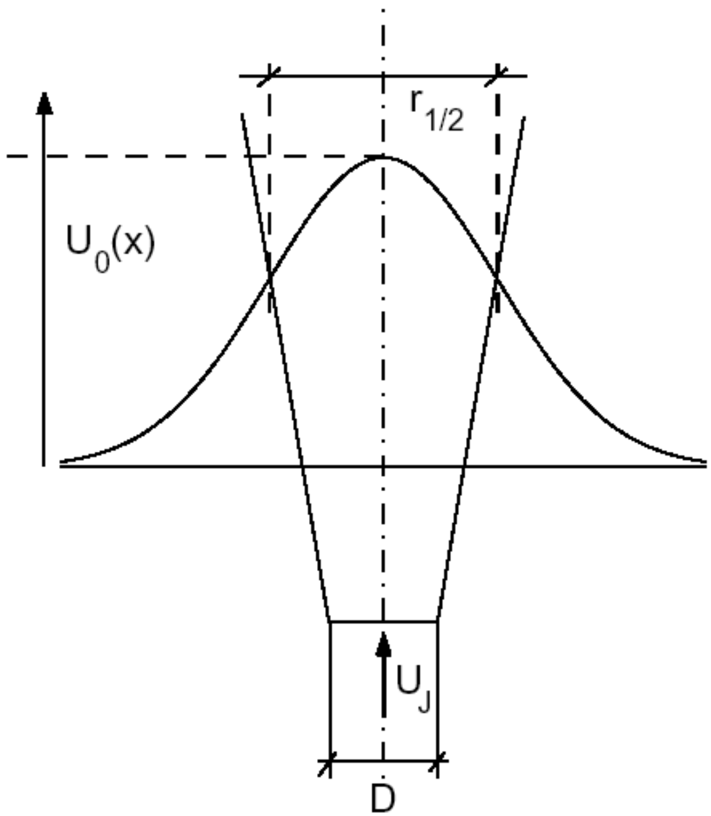


Free Shear Flows: Round Jet



The Round Jet

Assumptions: Statistically stationary and axisymmetric



Centerline velocity denoted as $U_0(x)$

Jet half width $r_{1/2}$ is defined as:

$$\langle U(x, r_{1/2}(x))\rangle = \frac{1}{2}U_0(x)$$

Similarity:

$$\frac{\langle U(\boldsymbol{x})\rangle}{U_J} = f\left(\frac{x}{D}, \frac{r}{D}, \mathrm{Re}\right)$$

The Round Jet

For large Re:

$$\frac{\langle U(\boldsymbol{x})\rangle}{U_J} = f\left(\frac{x}{D}, \frac{r}{D}\right)$$

Since jet half width and centerline velocity have no r-dependence:

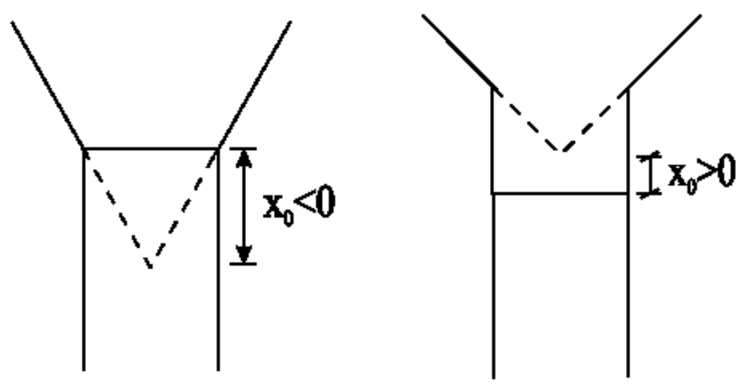
$$\frac{r_{1/2}(x)}{D} = f_{r_{1/2}}\left(\frac{x}{D}\right)$$

$$\frac{U_0(x)}{U_J} = f_{u_0}\left(\frac{x}{D}\right)$$

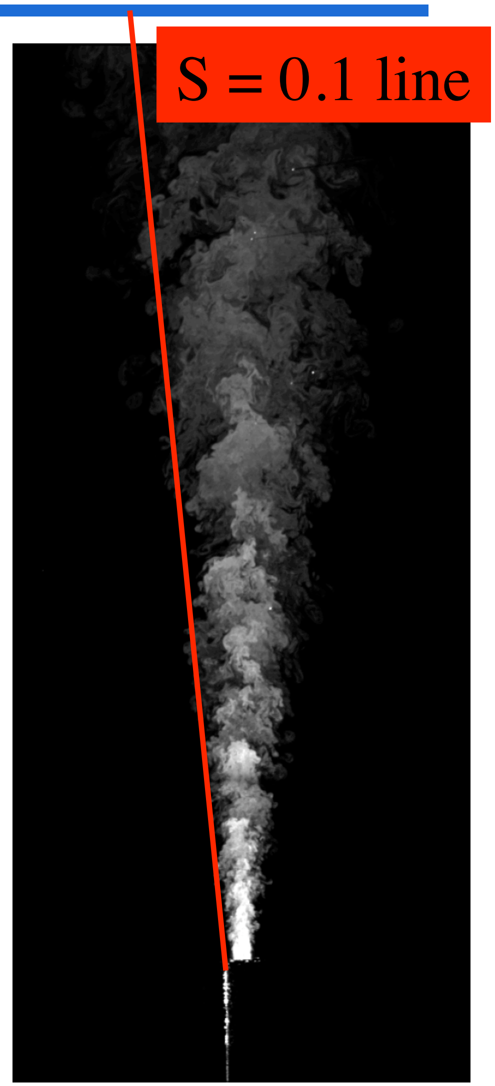
The Round Jet

$$\Rightarrow \qquad \frac{d\,r_{1/2}}{d\,x} = \text{const} = S$$

$$\Rightarrow \qquad r_{1/2} = S(x - x_0) \qquad \text{Exp:} \quad S \approx 0.1$$

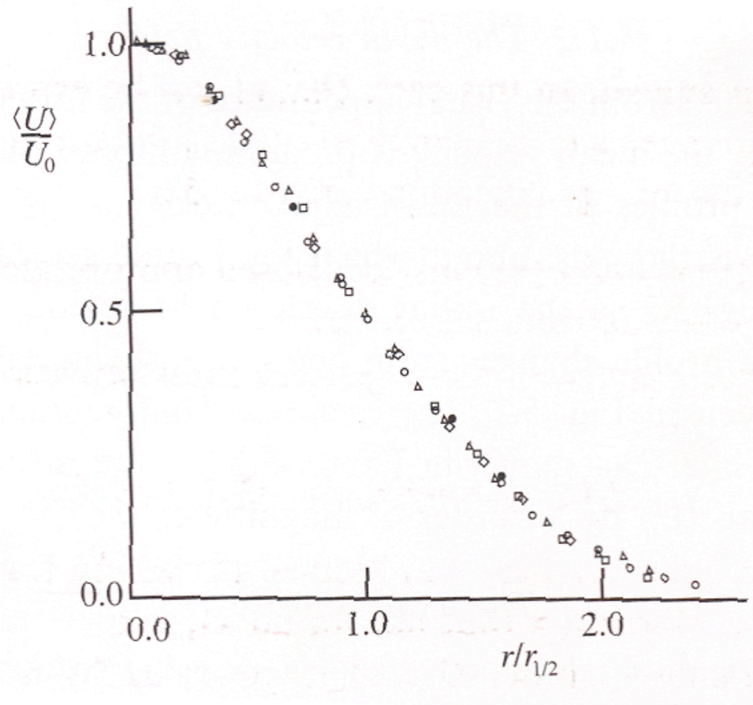


x_0 is virtual origin of jet



The Round Jet

Self-Similarity



Experimental data for various x/D

Radial similarity coordinates

$$\frac{\langle U \rangle}{U_0} = f_\xi\left(\frac{r}{r_{1/2}}\right) = f_\xi(\xi) = f_\eta\left(\frac{r}{x - x_0}\right) = f_\eta(\eta)$$

with

$$\eta = S\xi \quad (S \approx 0.1)$$

and ξ and η independent of x

The Round Jet

Momentum Balance

$$I(x) = 2\pi \int_0^\infty \rho \langle U(x,r) \rangle^2 \; r\,dr = 2\pi \rho U_0^2 r_{1/2}^2 \underbrace{\int_0^\infty f_\xi(\xi)\xi \, d\xi}_{\neq f(x)} = \text{const.}$$

$$\Rightarrow \qquad U_0^2(x)\, r_{1/2}^2 \;=\; \text{const.}$$

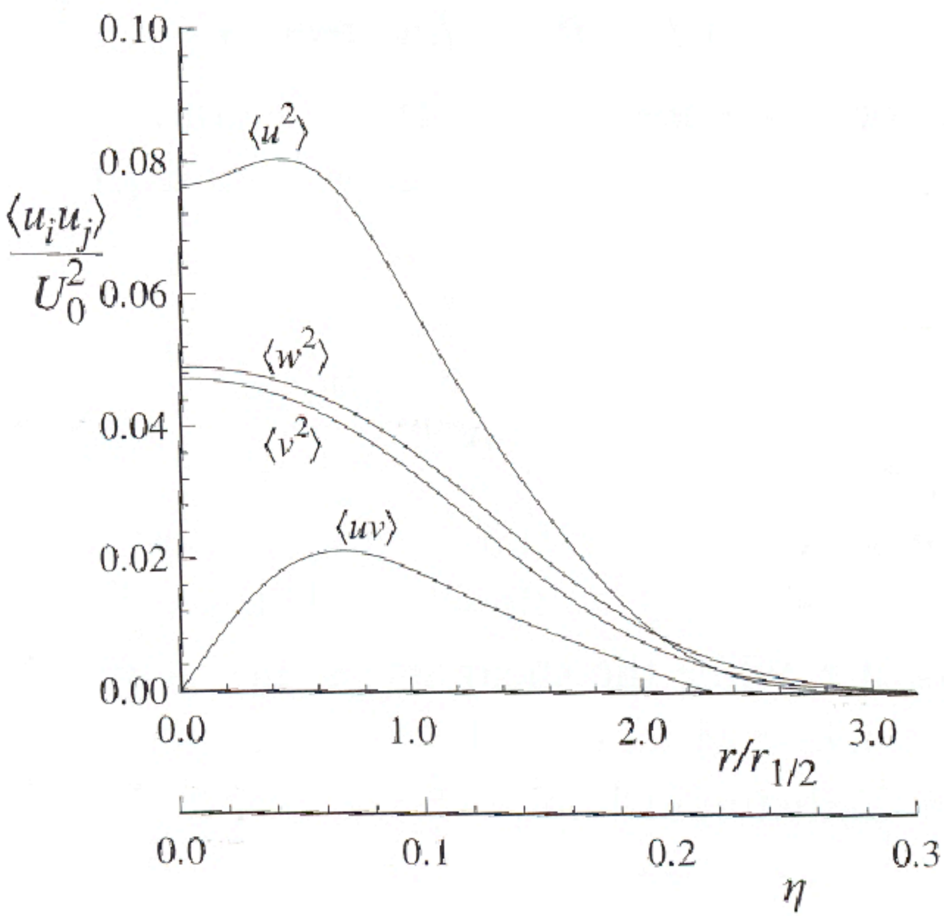
$$\Rightarrow \qquad U_0 \sim \frac{1}{r_{1/2}}$$

$$\frac{U_0(x)}{U_J} = \frac{\text{const}}{r_{1/2}/D} = \frac{BD}{x - x_0}$$

$$\Rightarrow \qquad \text{Re}_{\text{local}} = \frac{U_0(x) \cdot r_{1/2}(x)}{\nu} = \text{const.}$$

The Round Jet

Reynolds Stresses are self-similar



Reynolds stresses from experiments in a round jet

The Round Jet

Take mean

$$\underbrace{\frac{\partial k}{\partial t} + \langle U_i \rangle \frac{\partial k}{\partial x_i}}_{SDM} + \underbrace{\frac{\partial}{\partial x_i}\left[\frac{1}{2}\langle u_i u_j u_j \rangle + \frac{1}{\rho}\langle u_i p \rangle - 2\nu \langle u_j s_{ij} \rangle\right]}_{T}$$

$$= \underbrace{- \langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i}}_{\mathcal{P}} - \underbrace{\frac{\nu}{2}\left\langle \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 \right\rangle}_{\varepsilon = 2\nu \langle s_{ij} s_{ij} \rangle}$$

SDM: Substantial derivative of the mean
T: Flux term. Only describes transport

The Round Jet

- Example: Production in turbulent jet mostly in high shear regions, but non-zero at centerline
- Equation for kinetic energy of mean flow $\overline{E} = \frac{1}{2}\langle U_i \rangle^2$
  $\langle$Mean momentum equation $\cdot \langle U_j \rangle\rangle$

$$\frac{\partial \overline{E}}{\partial t} + \langle U_i \rangle \frac{\partial \overline{E}}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\langle U_j \rangle \langle u_i u_j \rangle + \frac{1}{\rho}\langle U_i \rangle \langle P \rangle - 2\nu \langle U_j \rangle \overline{S_{ij}} \right)$$

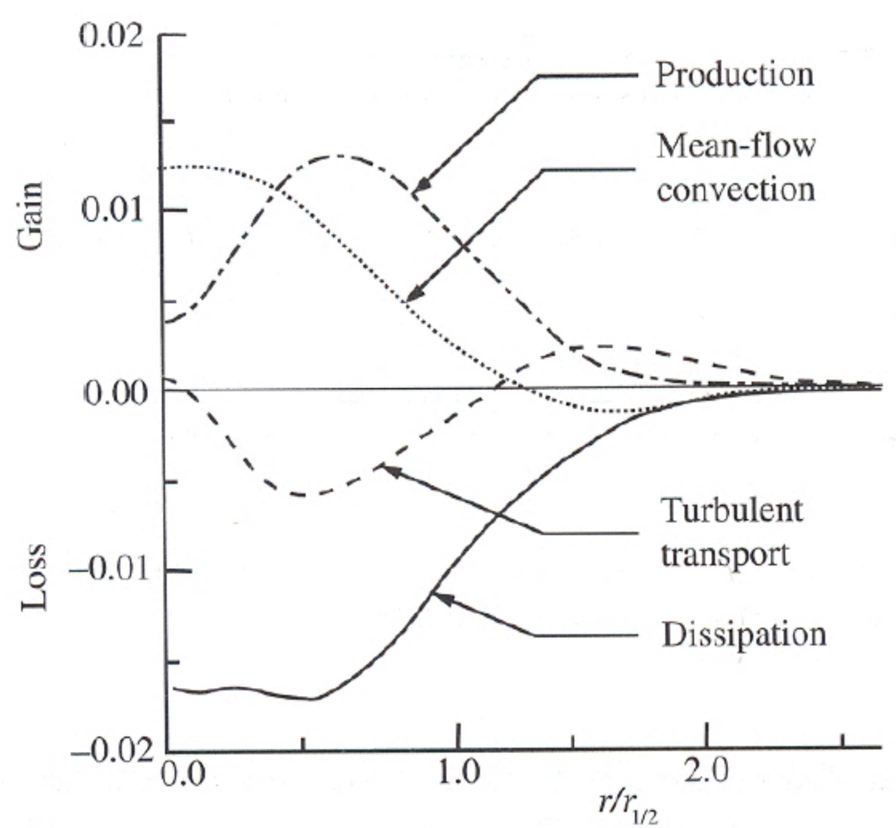
$$= \underbrace{\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i}}_{-P} - 2\nu \overline{S}_{ij}\, \overline{S}_{ij}$$

  - Turbulent kinetic energy production appears as sink for mean flow kinetic energy
  - Turbulence draws energy from mean flow

- Production occurs on large scales and leads to large turbulent structures

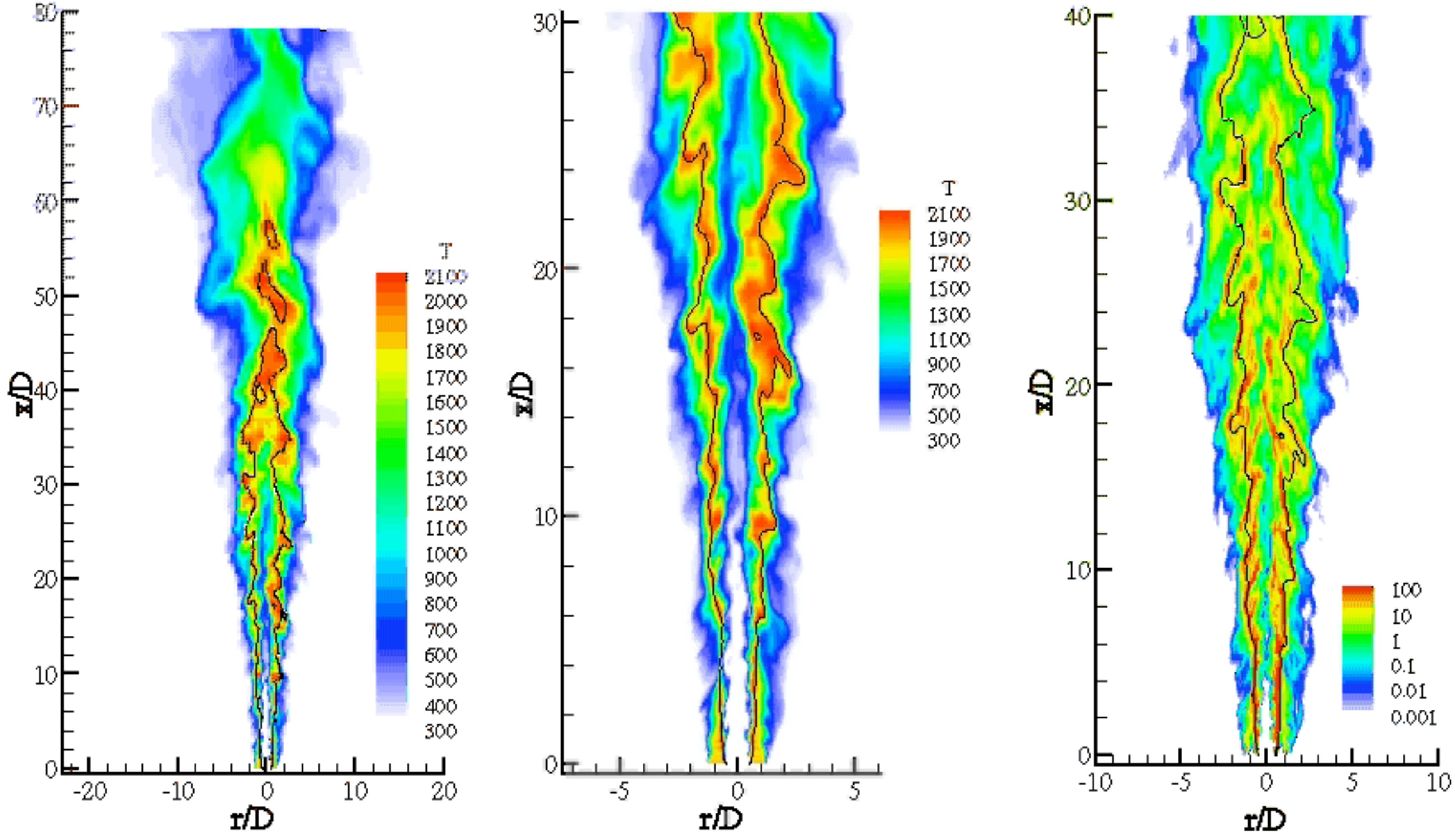
The Round Jet

Budget of Kinetic Energy



$\mathcal{P}$ non-zero at the centerline!

The Round Jet



Turbulent Mixing Layer

- Flow between two uniform parallel streams of different velocities $U_h > U_l \geq 0$

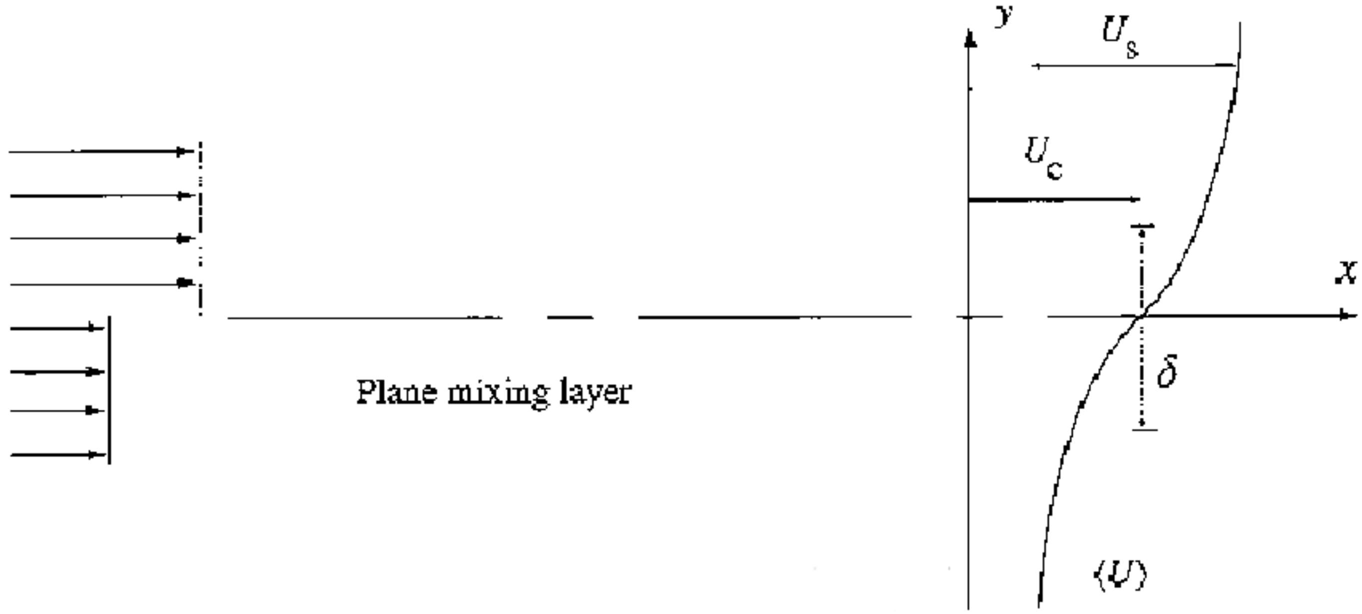


Fig. 5.14b from Pope's Turbulent Flows book

Turbulent Mixing Layer

- Two velocities $\longrightarrow$ new parameter U_l/U_h

$$\frac{\langle U(\boldsymbol{x},t)\rangle}{U_h} = f\left(\mathrm{Re}, \frac{y}{x}, \frac{U_l}{U_h}\right)$$

$\longrightarrow$ mixing layer is self-similar

- Characteristic velocity

$$U_c \equiv \frac{1}{2}(U_h + U_l)$$

- Velocity difference

$$U_s \equiv U_h - U_l$$

Turbulent Mixing Layer

- Characteristic mixing layer width

  With $y_\alpha(x)$ defined by $\langle U(x, y_\alpha(x)) \rangle = U_l + \alpha U_s$

$$\delta(x) = y_{0.9}(x) - y_{0.1}(x)$$

- Reference lateral position

$$\overline{y}(x) = \frac{1}{2}(y_{0.9}(x) + y_{0.1}(x))$$

Turbulent Mixing Layer

- Mixing layer is self-similar
- Scaled cross-stream coordinate

$$\xi = \frac{y - \overline{y}(x)}{\delta(x)}$$

- Scaled velocity

$$f(\xi) = \frac{\langle U \rangle - U_c}{U_s}$$

$$\Longrightarrow \quad f(\pm\infty) = \pm\frac{1}{2}$$

$$f(\pm\frac{1}{2}) = \pm 0.4$$

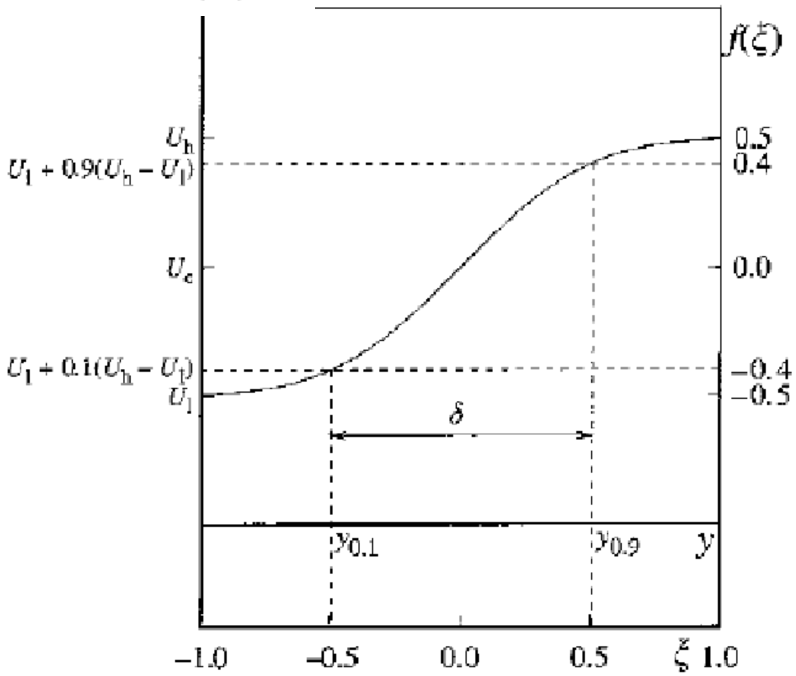


Fig. 5.21 from Pope's Turbulent Flows book

Turbulent Mixing Layer

- Flow not symmetric around $y = 0$

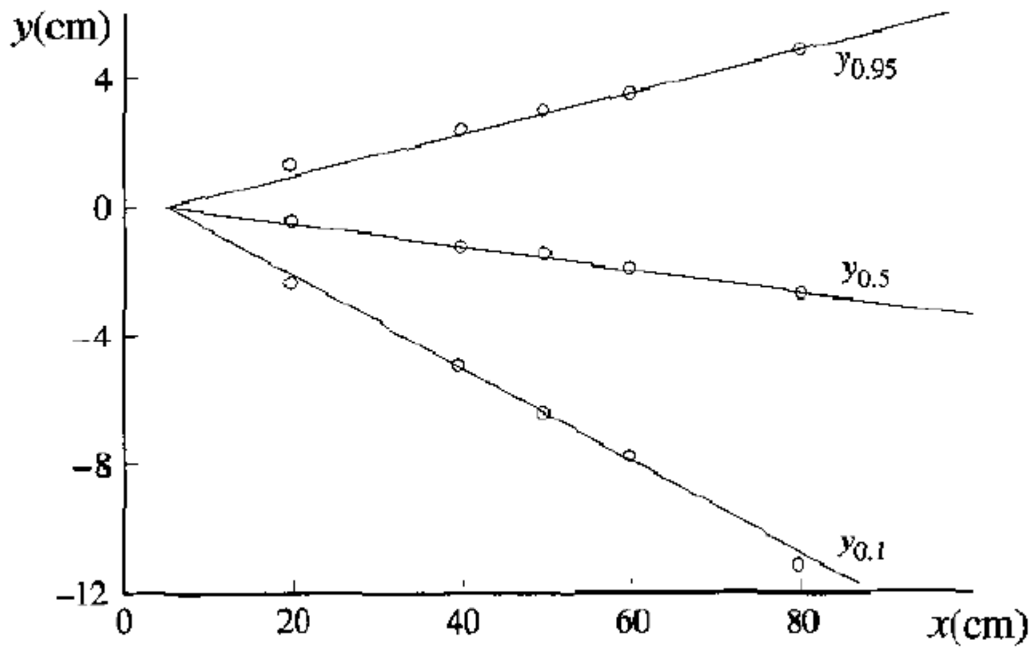
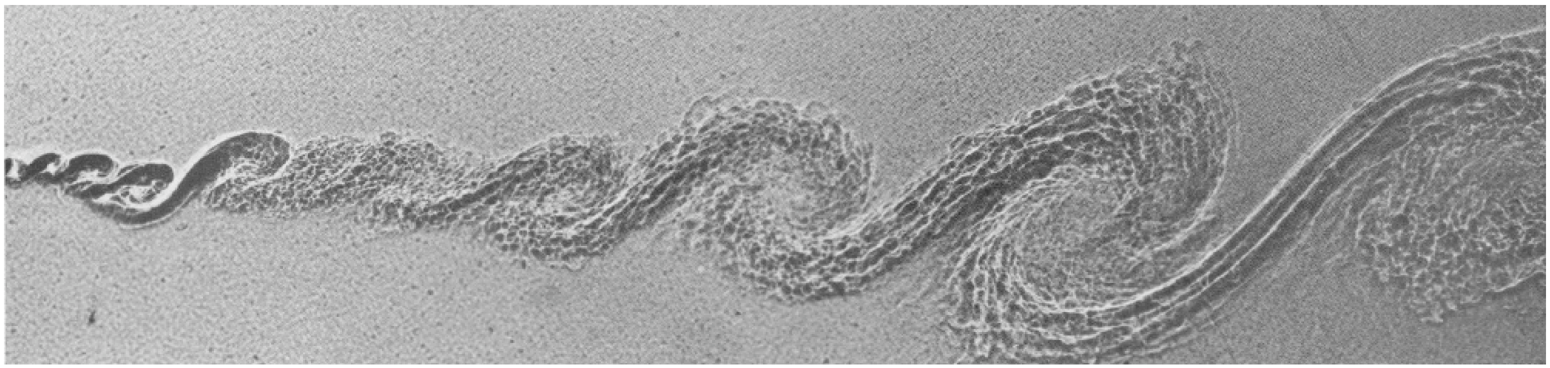


Fig. 5.23 from Pope's Turbulent Flows book

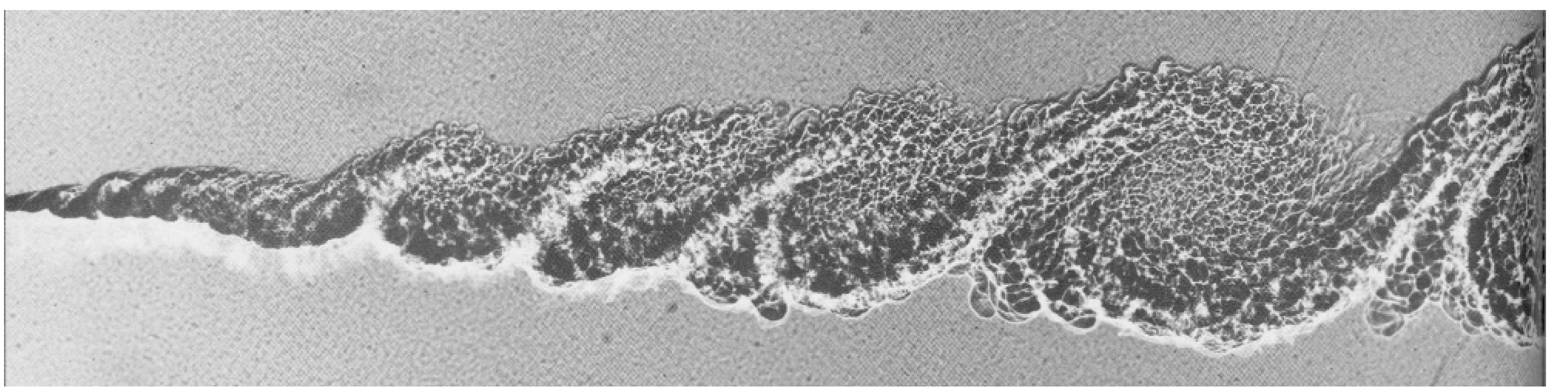
Turbulent Mixing Layer

Example: Visualization of turbulent mixing layer by Brown and Roshko

Low Reynolds number



High Reynolds number



Turbulent Mixing Layer

Temporal Mixing Layer

- Limit:

$$\frac{U_s}{U_c} \longrightarrow 0$$

- Boundary layer equation

$$U_c \frac{\partial \langle U \rangle}{\partial x} = -\frac{\partial \langle uv \rangle}{\partial y}$$

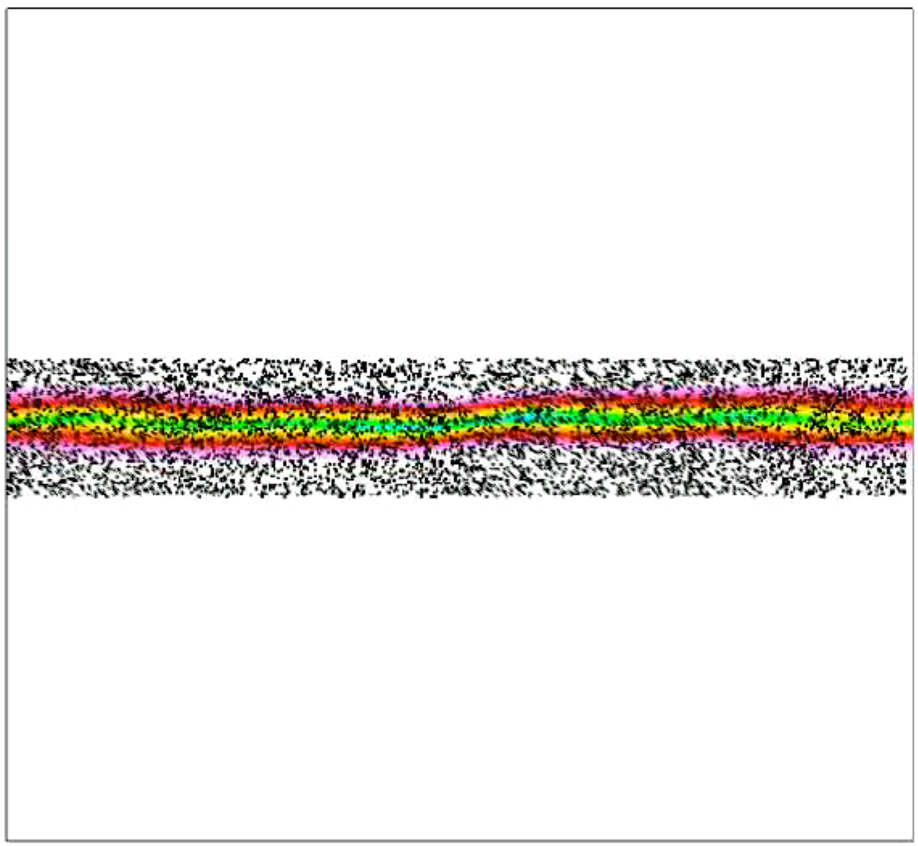
With $\tau = x/U_c$

$$\frac{\partial \langle U \rangle}{\partial \tau} = -\frac{\partial \langle uv \rangle}{\partial y}$$

- Assumption of statistical homogeneity in streamwise and spanwise direction leads to same equation

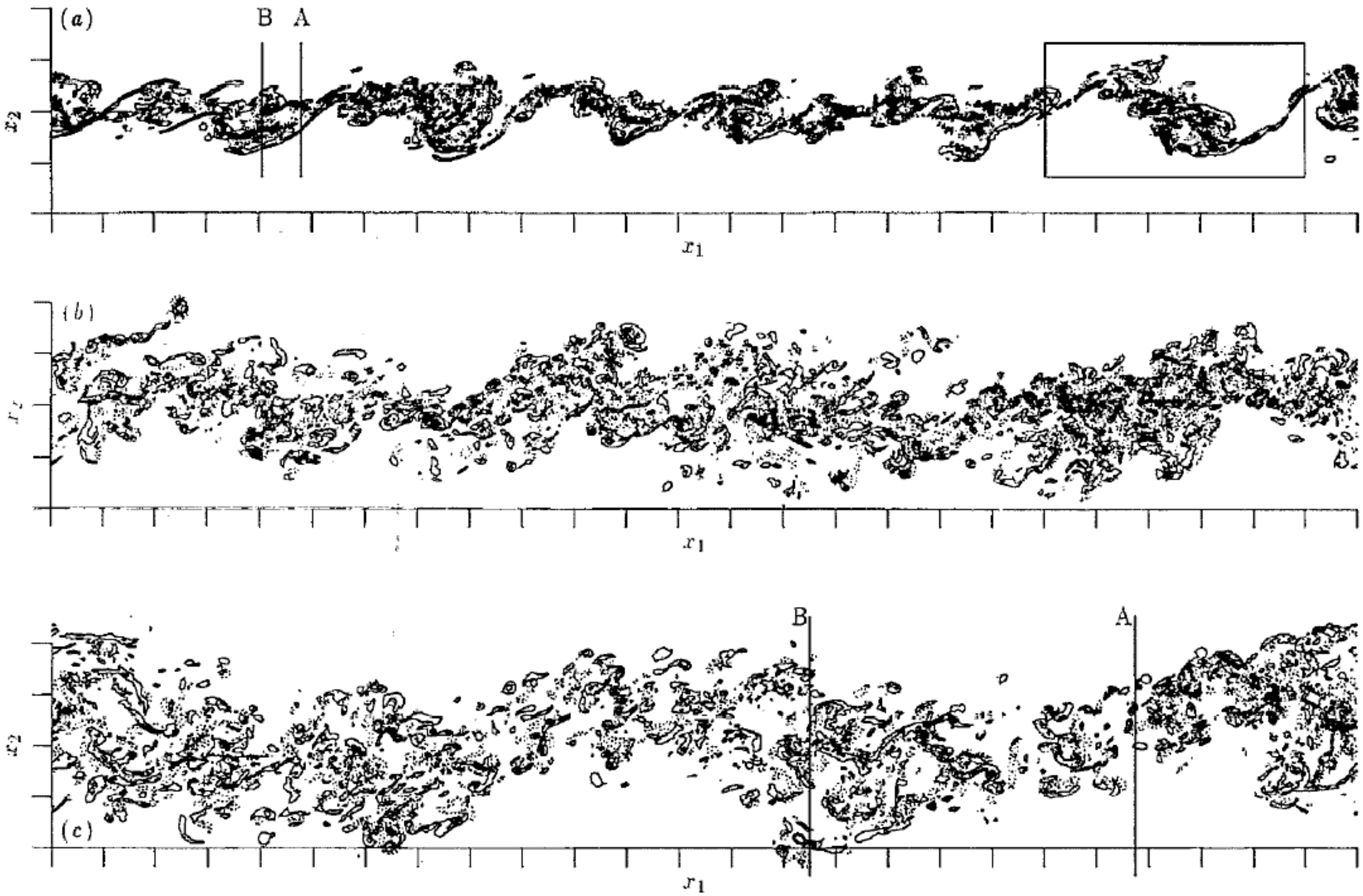
Turbulent Mixing Layer

Example: DNS of particle-laden temporally evolving mixing layer



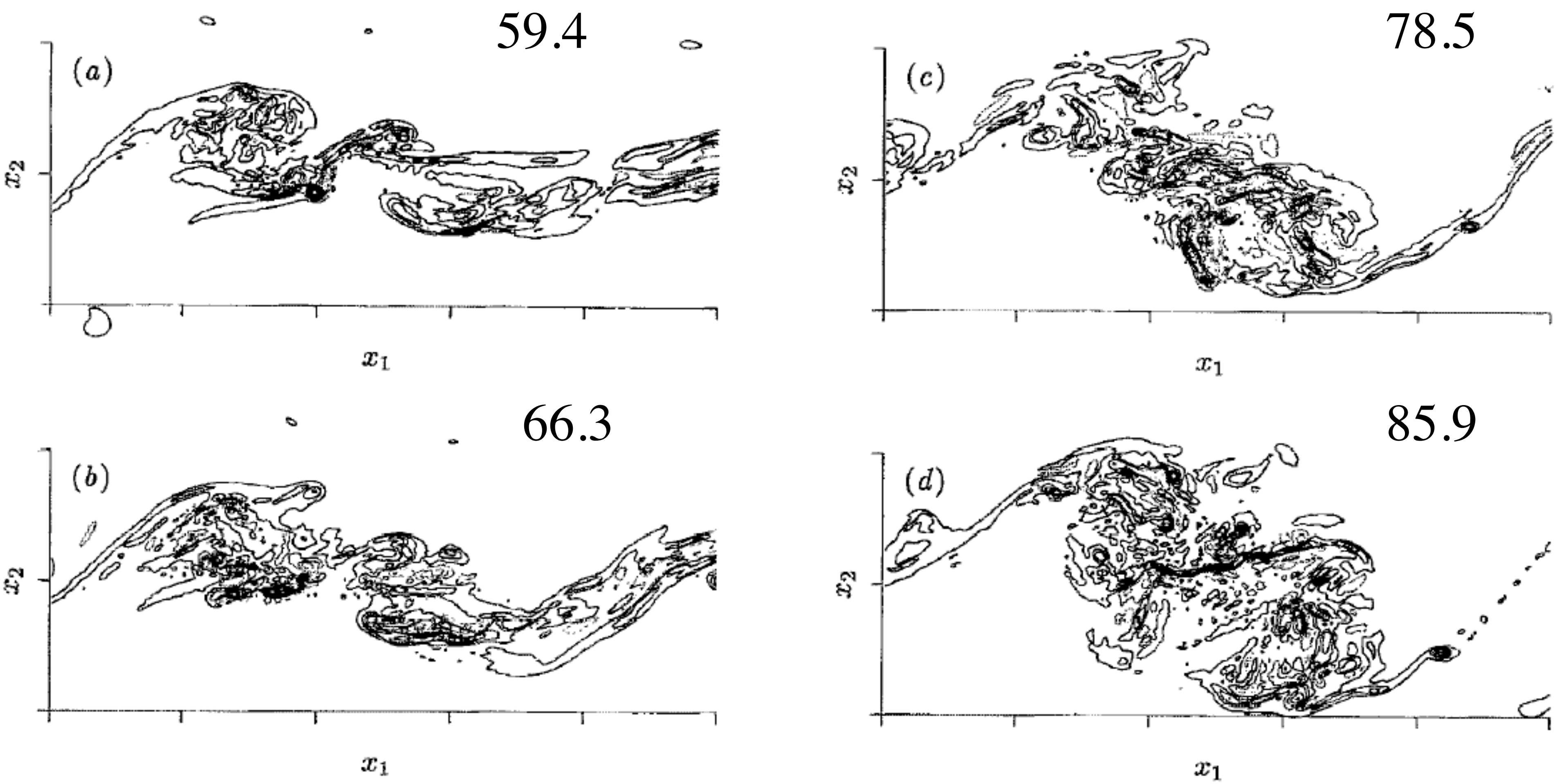
Turbulent Mixing Layer

DNS of Rogers and Moser (1994): Spanwise vorticity



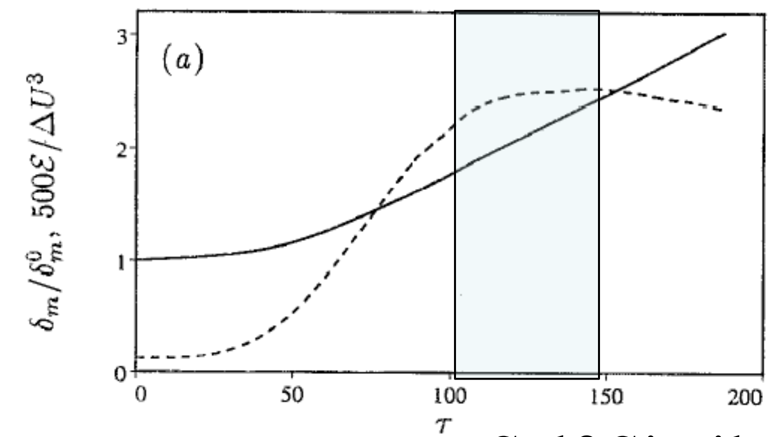
Turbulent Mixing Layer

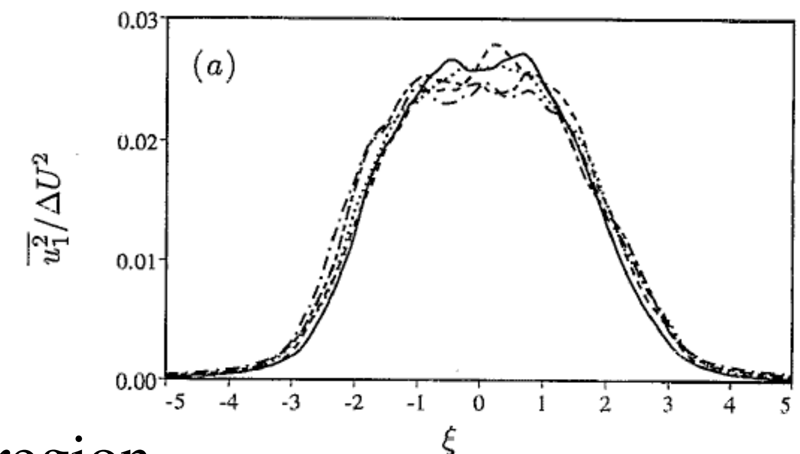
DNS of Rogers and Moser (1994): Spanwise vorticity



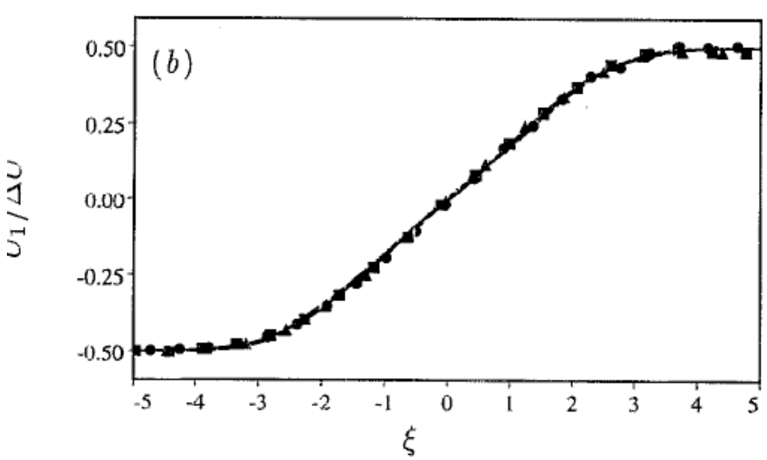
Turbulent Mixing Layer

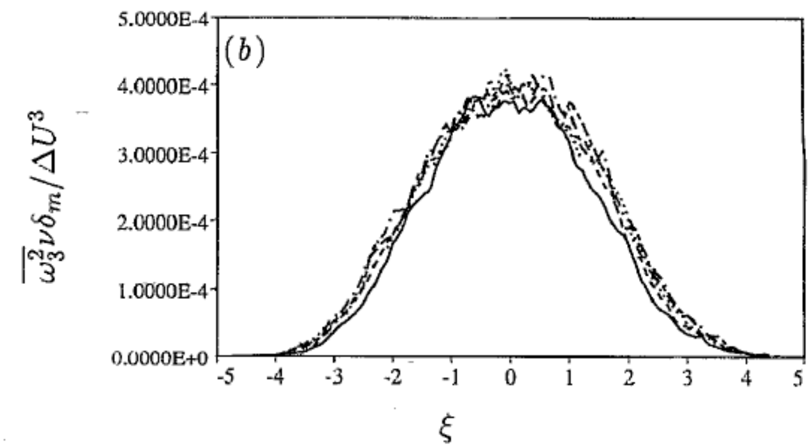
DNS of Rogers and Moser (1994): Self-Similarity





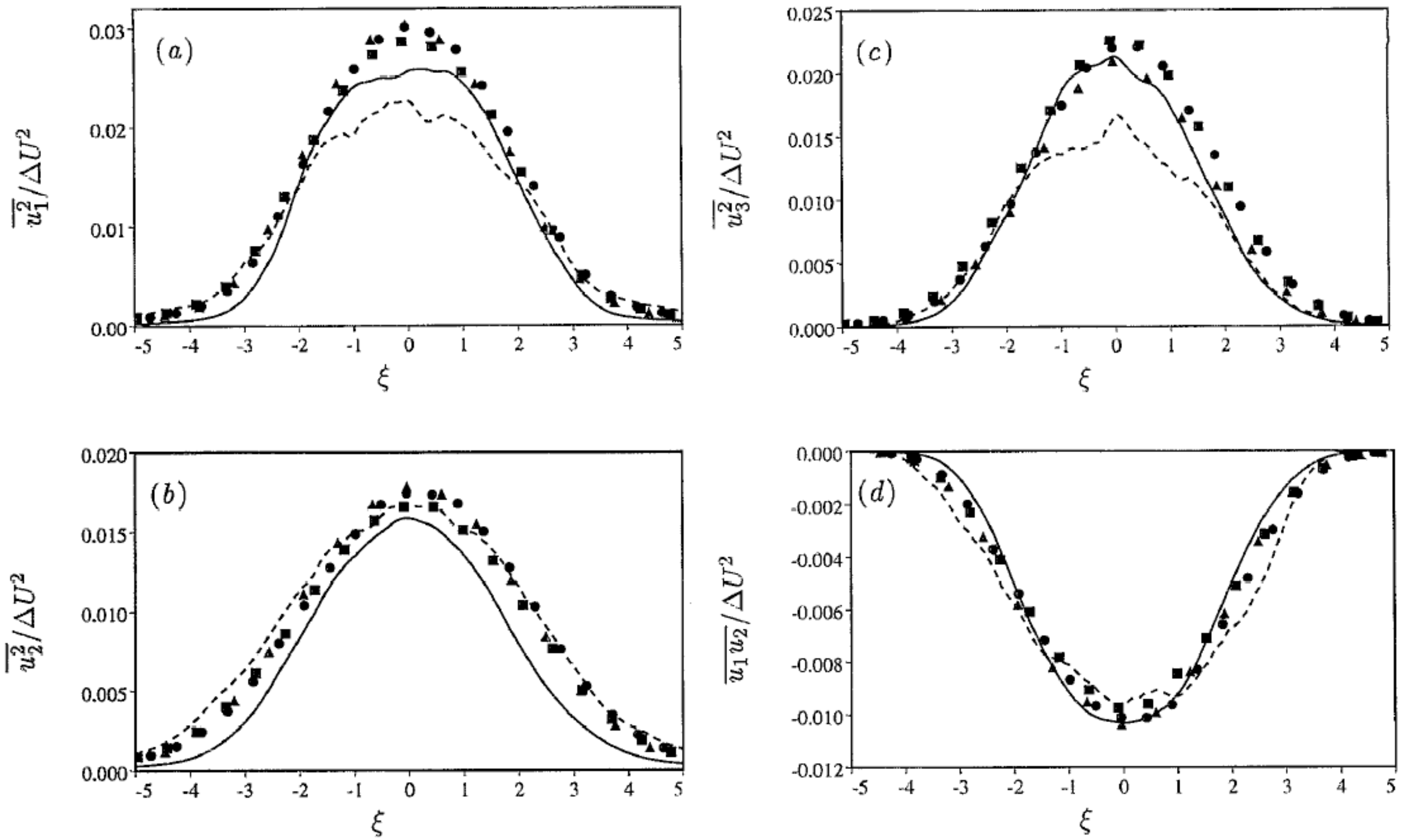
Self Similar region





Turbulent Mixing Layer

DNS of Rogers and Moser (1994): Reynolds Stress Tensor



Turbulent Mixing Layer

Spreading Rate

$$\frac{d\delta(x)}{dt} = \text{const} = \frac{U_s}{U_c} S \qquad (S \approx 0.06 - 0.11)$$

$$\Longrightarrow S = \frac{U_c}{U_s} \frac{d\delta(x)}{dt} \qquad \left(\text{independent of } \frac{U_s}{U_c} \right)$$

Turbulent Mixing Layer

Turbulent Kinetic Energy Flow Rate

$$K(x) \sim \int_{-\infty}^{\infty} \langle U \rangle \, k dy \sim U_c U_s^2 \delta \sim x$$

$k(x)$ increasing with x

$$\Longrightarrow \quad \mathcal{P} > \varepsilon \quad \left(\text{Rogers, Moser (1994): } \frac{\mathcal{P}}{\varepsilon} = 1.4 \right)$$

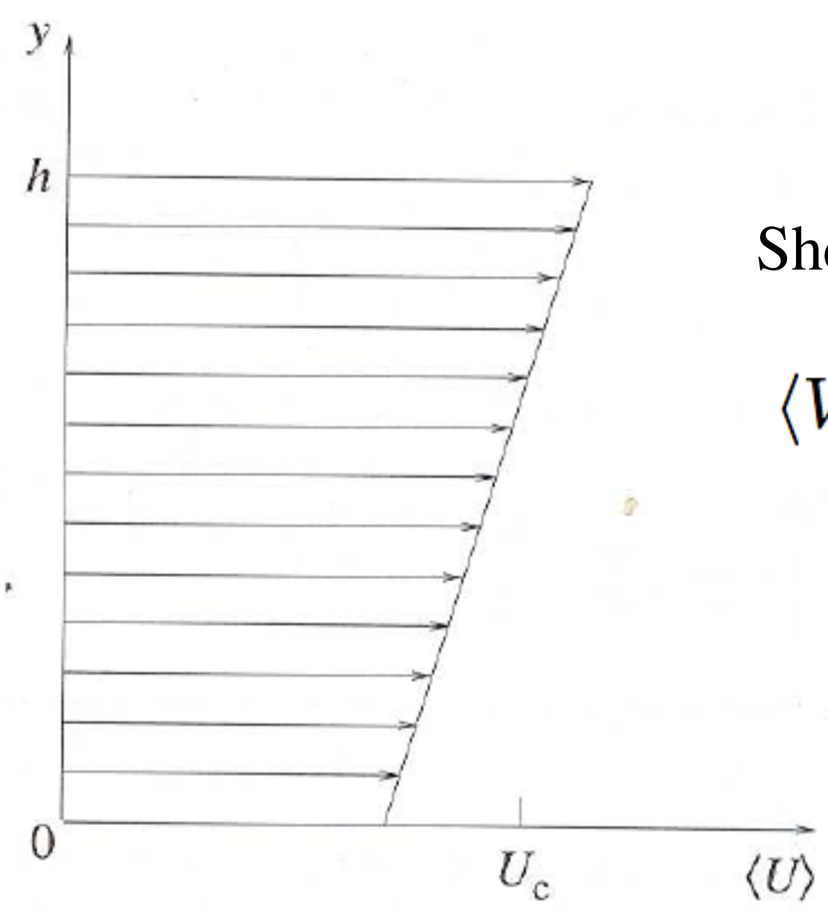
Homogeneous and Isotropic Turbulence

Homogeneous Turbulence: *Turbulence in which the statistics of fluctuating components of velocity* $\mathbf{u}(\mathbf{x},t)$ *and pressure* $p'(\mathbf{x},t)$ *are invariant under translation in space*

The imposed mean velocity gradient $\partial \langle U_i \rangle / \partial x_j$ must be uniform in space, but can vary in time

Here we examine homogeneous shear flow with imposed mean velocity gradient, $\mathcal{S} = \partial \langle U \rangle / \partial y = \text{const}$

Homogeneous and Isotropic Turbulence



Shear Rate S = const

$$\langle V \rangle = \langle W \rangle = 0$$

Homogeneous and Isotropic Turbulence

Comparison with Temporal Mixing Layer

- While the center of the temperal mixing layer appears to have reasonably uniform mean shear rate, $\mathcal{S} = \partial \langle U \rangle / \partial x = \text{const.}$, Reynolds stresses exhibit spatial variations
- At the centerline of TML, the TKE does not change with time, yet $\mathcal{P}/\varepsilon \approx 1.4$
- TKE is transported away from center of mixing layer. This transport requires statistical inhomogeneity
- Homogeneous turbulence is statistically homogeneous and therefore does not include a tranport term in TKE equation

Homogeneous and Isotropic Turbulence

TKE Equation for Homogeneous Turbulence

$$\begin{aligned}
\frac{\partial k}{\partial t} + \langle U_j \rangle \frac{\partial k}{\partial x_j} &= -\frac{\partial T'}{\partial x_j} + \mathcal{P} - \varepsilon \\
\mathcal{P} &\equiv -\langle u_i u_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} \\
T'_i &\equiv \frac{1}{2} \langle u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle \\
s_{ij} &\equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\end{aligned}$$

NOTE: *Because u_j and p' are statistically homogeneous, the mean of any combination of these quantities will have no spatial gradients (e.g. $\partial \langle u_i p' \rangle / \partial x_j = 0$)*

Homogeneous and Isotropic Turbulence

- Consider homogeneous turbulence field and assess if inhomogeneities in k could arise in Eq. 1
- Take closer look at turbulent transport term $(-\partial T'/\partial x_j)$ of TKE equation (Eqn. 1)
  - From Eq. 3, it can be seen that all three terms on the right hand side are simply averages of combinations of u_j and p' multiplied by constants
  - It is clear that since all these terms are constant in space, their gradients are zero. Hence, $-\partial T'/\partial x_j = 0$
- Similarly, the convection term will be zero as k is also constant in space for homogeneous turbulence. Therefore $\partial k/\partial x_i = 0$
- Therefore for homogeneous turbulence, the turbulent kinetic energy equation reduces to

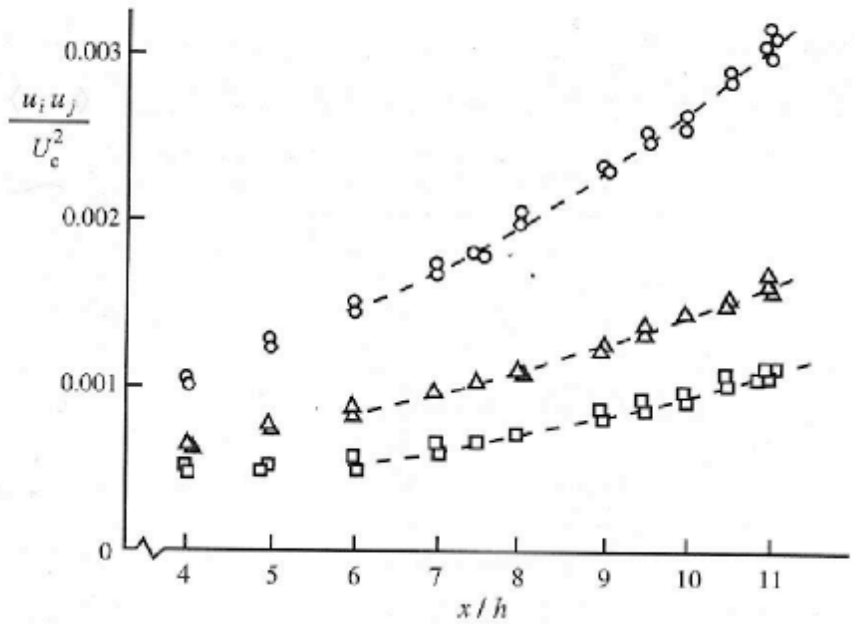
$$\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{P} - \varepsilon \tag{5}$$

Homogeneous and Isotropic Turbulence

- The term $\mathcal{P}$ is non-zero, but because $\mathcal{S} = \text{const}$, the term is constant in space
- Also ε is depends only on turbulent fluctuations and is therefore constant in space
- According to Eq. 5 the evolution of k is therefore the same everywhere in space and the turbulence remains homogeneous

Homogeneous and Isotropic Turbulence

Self-Similarity of Homogeneous Shear Flow



	Tavoularis and Corrsin		Rogers and Moin
	$x/h = 7.5$	$x/h = 11.0$	$\mathcal{S}t = 8.0$
$\langle u^2 \rangle / k$	1.04	1.07	1.06
$\langle v^2 \rangle / k$	0.37	0.37	0.32
$\langle w^2 \rangle / k$	0.58	0.56	0.62
$-\langle uv \rangle / k$	0.28	0.28	0.33
$-\rho_{uv}$	0.45	0.45	0.57
$\mathcal{S}k/\varepsilon$	6.5	6.1	4.3
$\mathcal{P}/\varepsilon$	1.8	1.7	1.4
$L_{11}\mathcal{S}/k^{1/2}$	4.0	4.0	3.7
$L_{11}/(k^{3/2}/\varepsilon)$	0.62	0.66	0.86

Homogeneous and Isotropic Turbulence

- After a developement time, homogeneous turbulent shear flow becomes self-similar
- Experimental data (Tavoularis and Corrsin, 1987) and DNS data (Rogers and Moin, 1987) provide insight into turb. homogeneous shear flows
- Exp. and DNS show that from $x/h = 7.5$ to $x/h = 11.0$ the TKE increases by 65 %, but normalized Reynolds stresses barely change at all
- Turbulent time-scale, $\tau = k/\varepsilon$, fixed by mean flow time-scale, $\mathcal{S}^{-1}$
- Longitudinal integral lengthscale, L_{11}, increases by 30 %, but is constant when scaled with $\mathcal{S}$ and k
- Production larger than dissipation, $\mathcal{P}/\varepsilon \approx 1.7$

Homogeneous and Isotropic Turbulence

Since $\tau = k/\varepsilon$, dividing Eq. 5 by ε, we get,

$$\frac{\tau}{k}\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\mathcal{P}}{\varepsilon} - 1. \tag{6}$$

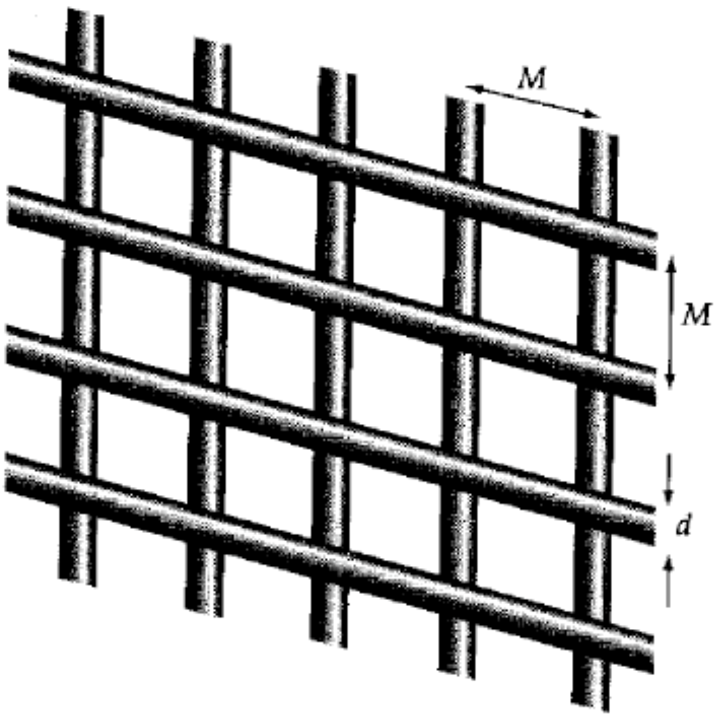
Considering that the turbulent time-scale $\tau \approx$ const, and the ratio of production to dissipation of TKE $\mathcal{P}/\varepsilon \approx 1.7$, the solution to Eq. 6 is

$$k(t) = k(0)\exp\left[\frac{t}{\tau}\left(\frac{\mathcal{P}}{\varepsilon} - 1\right)\right] \tag{7}$$

Since $\mathcal{P}/\varepsilon - 1 \approx 0.7 > 0$, the TKE will grow exponentially in time. Therefore, both ε and $L \equiv k^{3/2}/\varepsilon$ also grow exponentially in time.

Homogeneous and Isotropic Turbulence

Grid Turbulence – Isotropic Turbulence



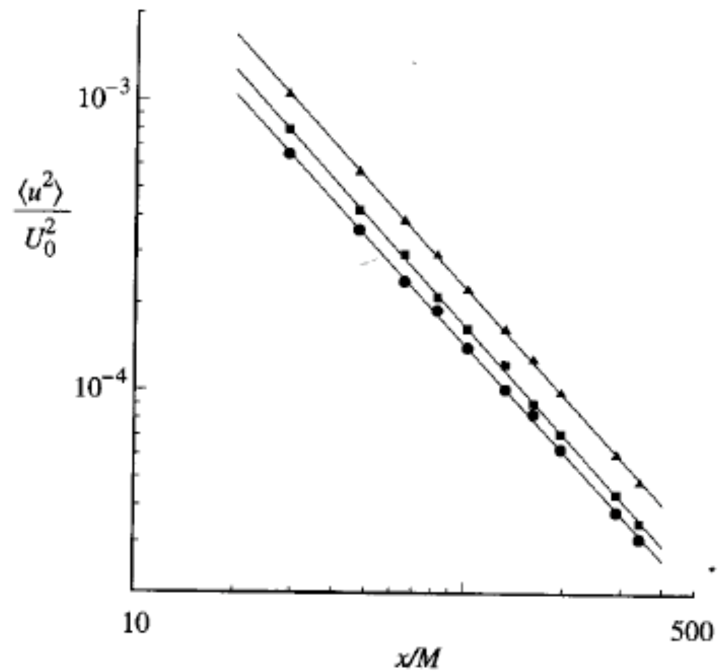


Fig. 5.32,33 A sketch of typical turbulence generating grid (left) and the decay of Reynolds stresses in grid turbulence (right). from Pope (2000) *Turbulent Flows*

Homogeneous and Isotropic Turbulence

- In absence of mean gradients ($\partial \langle U_i \rangle / \partial x_j = 0$), the production term in the TKE equation for homogeneous turbulence $\mathcal{P} \equiv -\langle u_i u_j \rangle \partial \langle U_i \rangle / \partial x_j = 0$
- Without production, the Turbulent Kinetic Energy decays with time
- This type of turbulence is referred to as *Homogeneous Isotropic Turbulence*

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\varepsilon \qquad (8)$$

- Isotropic turbulence can be approximated in wind-tunnel experiments by passing a uniform flow in the x-direction through a grid.

Homogeneous and Isotropic Turbulence

- Lab Frame: Flow is statistically stationary
- Moving Frame: Turbulence is approximately homogeneous and, using Taylor's hypothesis, evolves in time with $t = x/U_o$
- Symmetry implies that ideal experiments have statistically equivalent $\langle v^2 \rangle$ and $\langle w^2 \rangle$ with all shear stresses equal to zero
- However, $\langle u^2 \rangle$ is 10 % larger than $\langle v^2 \rangle$
- Experiments indicate that normal stresses and k decay in x following a power law

Isotropic Turbulence: A Few Essentials

- Reynolds Stress Tensor

$$\langle u_i u_j \rangle = \begin{pmatrix} \langle u^2 \rangle & 0 & 0 \\ 0 & \langle v^2 \rangle & 0 \\ 0 & 0 & \langle w^2 \rangle \end{pmatrix}$$

- Turbulent Kinetic Energy Equation

$$\frac{dk}{dt} = -\varepsilon$$

Homogeneous and Isotropic Turbulence

Lab Frame:

$$\frac{k}{U_o^2} = A\left(\frac{x - x_o}{M}\right)^{-n} \tag{9}$$

Where $1.15 < n < 1.45$, with $n = 1.3$ being consistent with nearly all experiments

Moving Frame:

$$k(t) = k_o\left(\frac{t}{t_o}\right)^{-n} \tag{10}$$

Given Eq. 8, decay of ε also follows a power law

$$\varepsilon(t) = \varepsilon_o\left(\frac{t}{t_o}\right)^{-(n+1)} \tag{11}$$

- Homogeneous Isotropic Turbulence is the simplest turbulent flow
- It has therefore been studied quite extensively
- Isotropic turbulence is different from typical applications as there is no turbulence production, but the importance of isotropic turbulence will become clear in the context of Kolmogorov's theory

Scales of Turbulent Motion

- Think about turbulence in wave number space
- Consider kinetic energy associated with different wave numbers or different length scales
- Kinetic energy content in range of wave numbers [κ_a, κ_b]

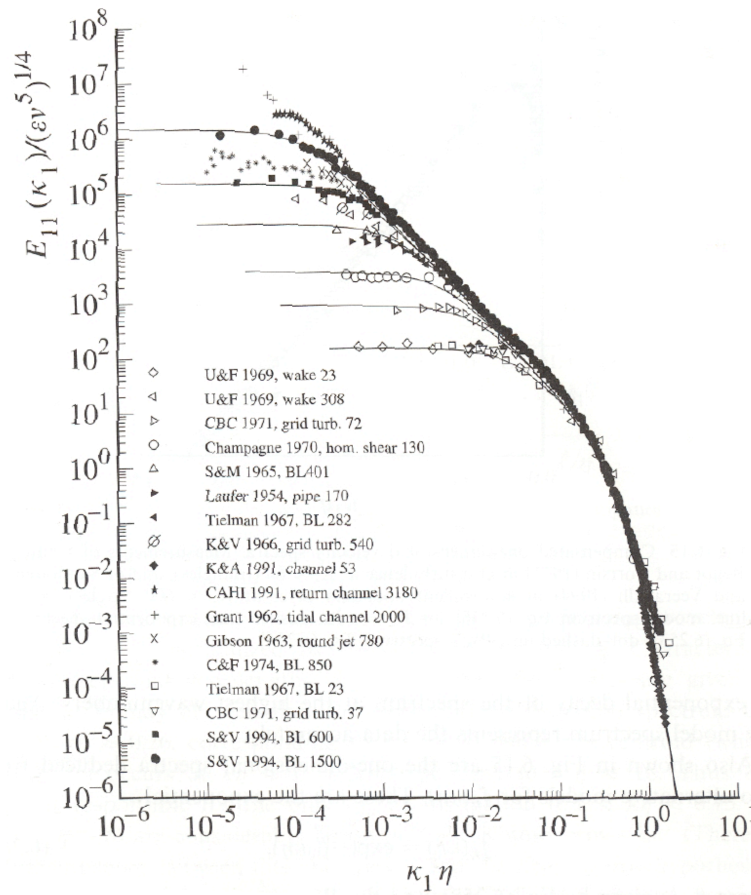
$$k(\kappa_a, \kappa_b) = \int_{\kappa_a}^{\kappa_b} E(\kappa)\, d\kappa$$

(Formal definition of energy spectrum will be given below)

Scales of Turbulent Motion

Energy spectrum used for

- Validate Kolmogorov Theory
- Assess dynamic importance of different scales
- Assess turbulence
- Assess Reynolds number
- Turbulence modeling



Scales of Turbulent Motion

Energy cascade:

- TKE produced at large scales
- Dissipation at smallest scales

$\longrightarrow$ Energy transfer from large to successively smaller scales

Richardson (1922):
Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity.

Scales of Turbulent Motion

Kolmogorov's hypothesis of local isotropy:

At high Re, small-scale turbulent motions ($L < L_{\mathrm{EI}}$) *are statistically isotropic.*

Evidence:

1. Anisotropy caused by large scales (geometry)
2. Small scales are result of large number of random events
3. Small scales have small time scales
   $\longrightarrow$ fast equilibrium

Scales of Turbulent Motion

Kolmogorov's first similarity hypothesis:

In every flow at high Re, the statistics of the small scale motions ($L < L_{\text{EI}}$) have universal form and are uniquely determined by ν and ε.

Reason:

- Consider scales $L < L_{\text{EI}}$
  $\longrightarrow$ small time scales lead to local stationarity
  $\longrightarrow$ local equilibrium

Scales of Turbulent Motion

Kolmogorov scales:

$$\left.\begin{aligned} \eta &= \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \\ u_\eta &= (\varepsilon\nu)^{1/4} \end{aligned}\right\} \quad \mathrm{Re}_\eta = 1$$

$$\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

Scales of Turbulent Motion

Consequence:

- Statistics of

$$\frac{\boldsymbol{u}}{u_\eta} = f(\nu, \varepsilon)$$

- No non-dimensional group from ν and ε
- $\boldsymbol{u}/u_\eta$ is universal

Scales of Turbulent Motion

Kolmogorov's second similarity hypothesis:

In every turbulent flow at high Re, statistics of motions of scale L *in the range* $L_0 \gg L \gg \eta$ *have universal form uniquely determined by* ε, *independent of* ν

Random Processes, Random Fields, and Two-Point Correlations

Definition of Simple Statistical Measures for Random Processes

- Auto covariance

$$R(s) \equiv \langle u(t)u(t+s) \rangle$$

- Autocorrelation function

$$\rho(s) = \frac{\langle u(t)u(t+s) \rangle}{\langle u^2(t) \rangle}$$

Properties:

$$\begin{aligned} |\rho(s)| &\leq 1 \\ \rho(0) &= 1 \\ \rho(s) &= \rho(-s) \end{aligned}$$

Random Processes, Random Fields, and Two-Point Correlations

- Integral time-scale

$$\overline{\tau} = \int_0^\infty \rho(s)ds$$

- Frequency spectrum is Fourier transform of (twice) the autocovariance

$$E(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(s)e^{-i\omega s}ds = \frac{2}{\pi}\int_0^{\infty} R(s)e^{-i\omega s}ds$$

Contribution to variance from frequency range $\omega_a \leq \omega \leq \omega_b$ given by

$$\int_{\omega_a}^{\omega_b} E(\omega)d\omega$$

Random Processes, Random Fields, and Two-Point Correlations

Two-Point Correlations

Two-point correlations of random velocity field

$$R_{ij}(\boldsymbol{r},\boldsymbol{x},t) \equiv \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x}+\boldsymbol{r},t)\rangle$$

Integral length scales, e.g.

$$L_{11} = \frac{1}{R_{11}(0,\boldsymbol{x},t)}\int_0^\infty R_{11}(\boldsymbol{e_1}r,\boldsymbol{x},t)dr$$

Random Processes, Random Fields, and Two-Point Correlations

Velocity spectrum tensor

- For homogeneous turbulence, Fourier transform of $R_{ij}(\boldsymbol{r}, t)$ is velocity spectrum tensor

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ij}(\boldsymbol{r}, t) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} dr_1 dr_2 dr_3$$

  $\boldsymbol{\kappa}$ is wave number vector with wave length $\lambda = 2\pi/|\boldsymbol{\kappa}|$

- Inverse transform

$$R_{ij}(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}, t) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\kappa_1 d\kappa_2 d\kappa_3$$

Random Processes, Random Fields, and Two-Point Correlations

Energy spectrum function

$$E(\kappa, t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ii}(\boldsymbol{\kappa}, t) \delta(|\boldsymbol{\kappa}| - \kappa) d\kappa_1 d\kappa_2 d\kappa_3$$

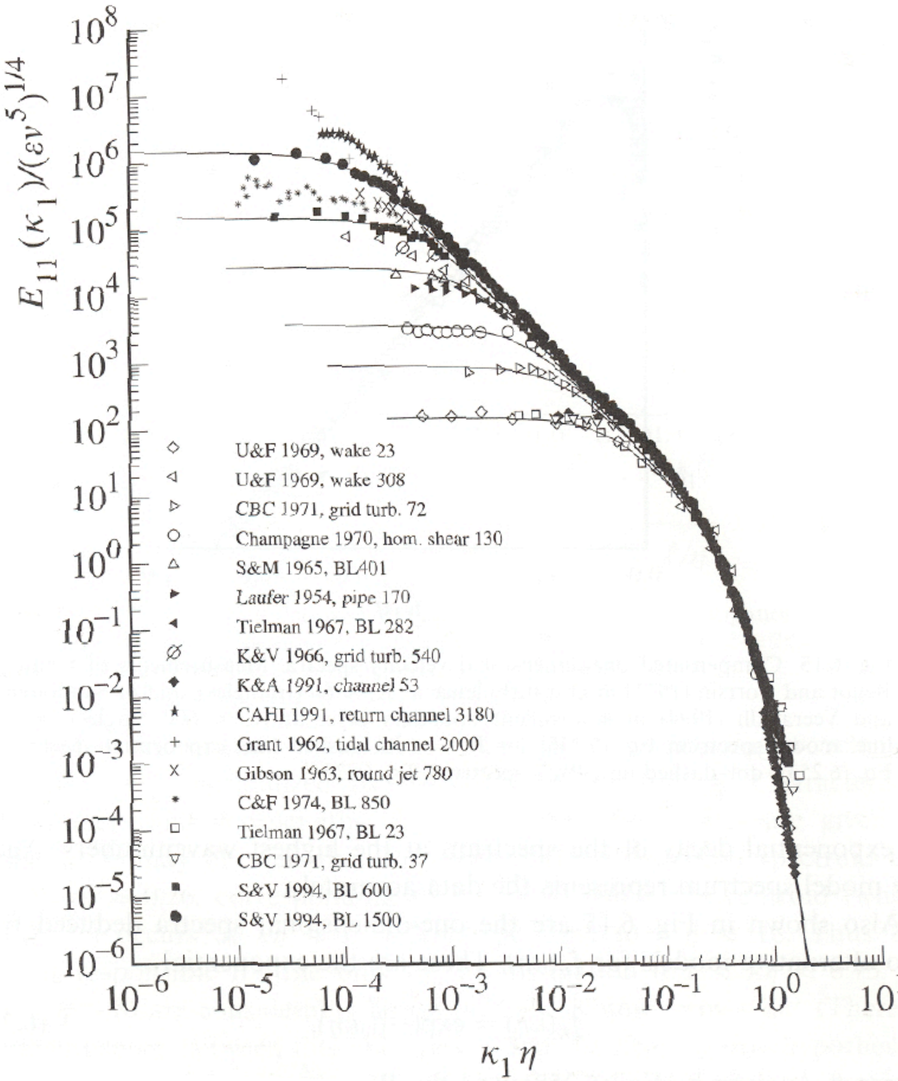
Turbulent kinetic energy

$$k = \frac{1}{2} \langle u_i^2 \rangle = \int_0^{\infty} E(\kappa) d\kappa$$

Scales of Turbulent Motion

Energy spectrum at different Reynolds numbers

- Confirms -5/3 scaling
- Universal for scales << L_0
- Confirms increasing intertial range for increasing Re



Scales of Turbulent Motion

Two-Point Correlation:

Consider homogenous isotropic turbulence:

$$R_{ij}(\boldsymbol{r},t) = \langle u_i(\boldsymbol{x},t)\, u_j(\boldsymbol{x}+\boldsymbol{r},t)\rangle$$

$$R_{ij}(0,t) = \langle u_i\, u_j\rangle \;=\; u'^2\delta_{ij} \quad \text{with} \quad u'^2 = u_1^2 = u_2^2 = u_3^2$$

TKE equation:

$$\frac{dk}{dt} = -\varepsilon$$

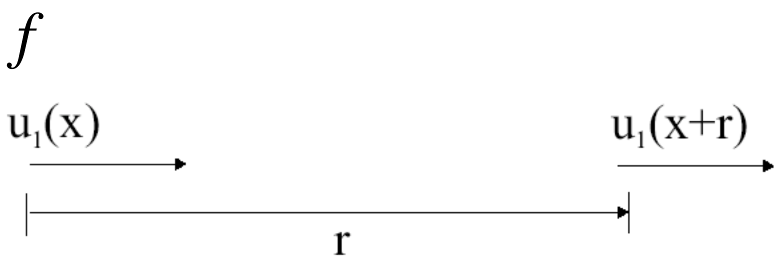
Scales of Turbulent Motion

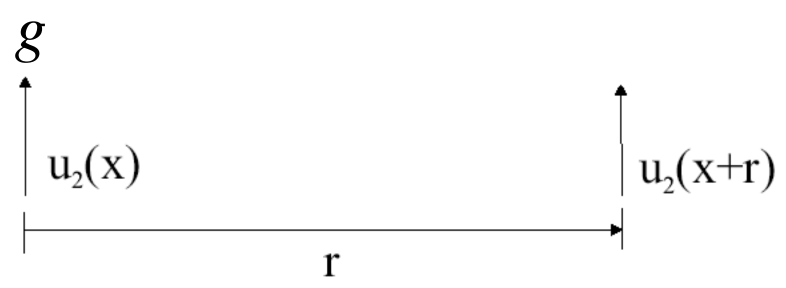
R_{ij} is an isotropic tensor, function of r_j:

$\Rightarrow$ only second order tensor that can be formed from r_j are δ_{ij} and $r_i r_j$.

$$R_{ij} = A(r) r_i r_j + B(r)\delta_{ij}$$

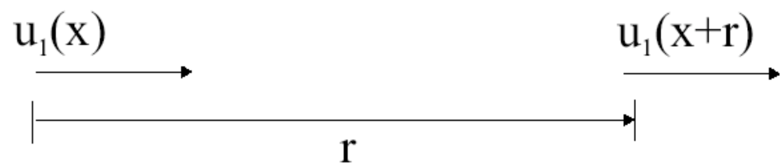
Determine A and B in terms of longitudinal and transverse autocorrelation f and g





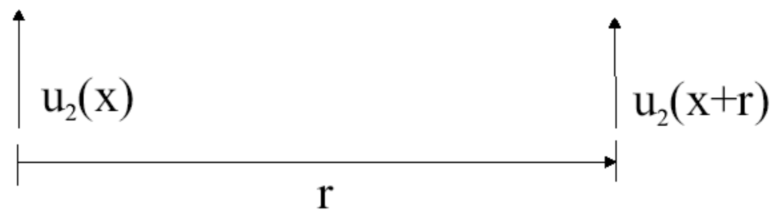
Scales of Turbulent Motion

Longitudinal autocorrelation in direction $\boldsymbol{r} = \boldsymbol{e_1} r = (r, 0, 0)^T$:



$$f = \frac{R_{11}}{u'^2} = \frac{1}{u'^2}(Ar^2 + B)$$

and transverse autocorrelation



$$g = \frac{R_{22}}{u'^2} = \frac{1}{u'^2} B$$

Therefore,

$$A = \frac{u'^2}{r^2}(f - g), \qquad B = u'^2 g$$

$$\Rightarrow \quad R_{ij} = u'^2 \left[(f - g)\frac{r_i r_j}{r^2} + g\delta_{ij} \right]$$

Scales of Turbulent Motion

From continuity

$$\frac{\partial R_{ij}}{\partial r_j} = 0$$

it follows

$$g(r) = f + \frac{r}{2} f'.$$

For homogeneous isotropic turbulence, R_{ij} can be expressed as a function of $f(r)$ only.

Scales of Turbulent Motion

Integral length scale

Longitudinal:

$$L_{11}(t) = \int_0^\infty f(r,t)dr$$

Transverse:

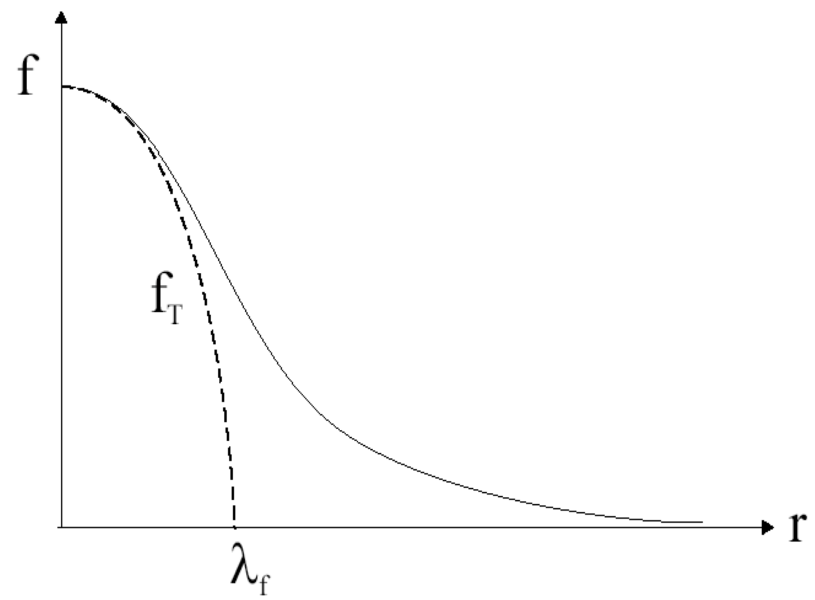
$$L_{22}(t) = \int_0^\infty g(r,t)dr$$

Scales of Turbulent Motion

Taylor Microscale

Taylor series expansion of f for small r

$$\begin{aligned} f_T(r) &= f(0) + f'(0)\,r + \frac{1}{2} f''(0) r^2 + \cdots \\ &= 1 + \frac{1}{2} f''(0) r^2 \end{aligned}$$



Velocity Spectra

Two-Point Correlation and Velocity Spectrum Function

$$R_{ij}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \, d\boldsymbol{\kappa} \, ,$$

which formally is an inverse Fourier transform.

Hence

$$\Phi_{ij}(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ij}(\boldsymbol{r}) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \, d\boldsymbol{r}$$

Velocity Spectra

Velocity spectrum tensor represents

- Reynolds stresses for $r \longrightarrow 0$

$$\langle u_i u_j \rangle = R_{ij}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}) d\boldsymbol{\kappa}$$

- Dissipation rate

$$\varepsilon = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu \kappa^2 \Phi_{ii}(\boldsymbol{\kappa}) d\boldsymbol{\kappa}$$

Velocity Spectra

Energy Spectrum Function

Remove direction of velocity and wave number from velocity spectrum function

$$\Longrightarrow \qquad E(\kappa) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\Phi_{ii}(\boldsymbol{\kappa})\delta(|\boldsymbol{\kappa}| - \kappa)d\boldsymbol{\kappa}$$

Then

$$k = \int_0^{\infty} E(\kappa)d\kappa$$

and

$$\varepsilon = \int_0^{\infty} 2\nu\kappa^2 E(\kappa)d\kappa$$

Velocity Spectra

One-Dimensional Spectra

Spectrum in one direction $\boldsymbol{e_1}r$ as twice the one-dimensional Fourier transform of $R_{ij}(\boldsymbol{e_1}r)$

$$\Longrightarrow \quad E_{ij}(\kappa_1) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} R_{ij}(\boldsymbol{e_1}r)e^{-i\kappa_1 r_1}dr_1$$

Inverse

$$R_{ij}(\boldsymbol{e_1}r) = \frac{1}{2}\int_{-\infty}^{\infty} E_{ij}(\kappa_1)e^{i\kappa_1 r_1}d\kappa_1$$

Velocity Spectra

Example:

$$E_{11}(\kappa_1) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_{11}(\boldsymbol{e_1}r)e^{-i\kappa_1 r_1}dr_1$$

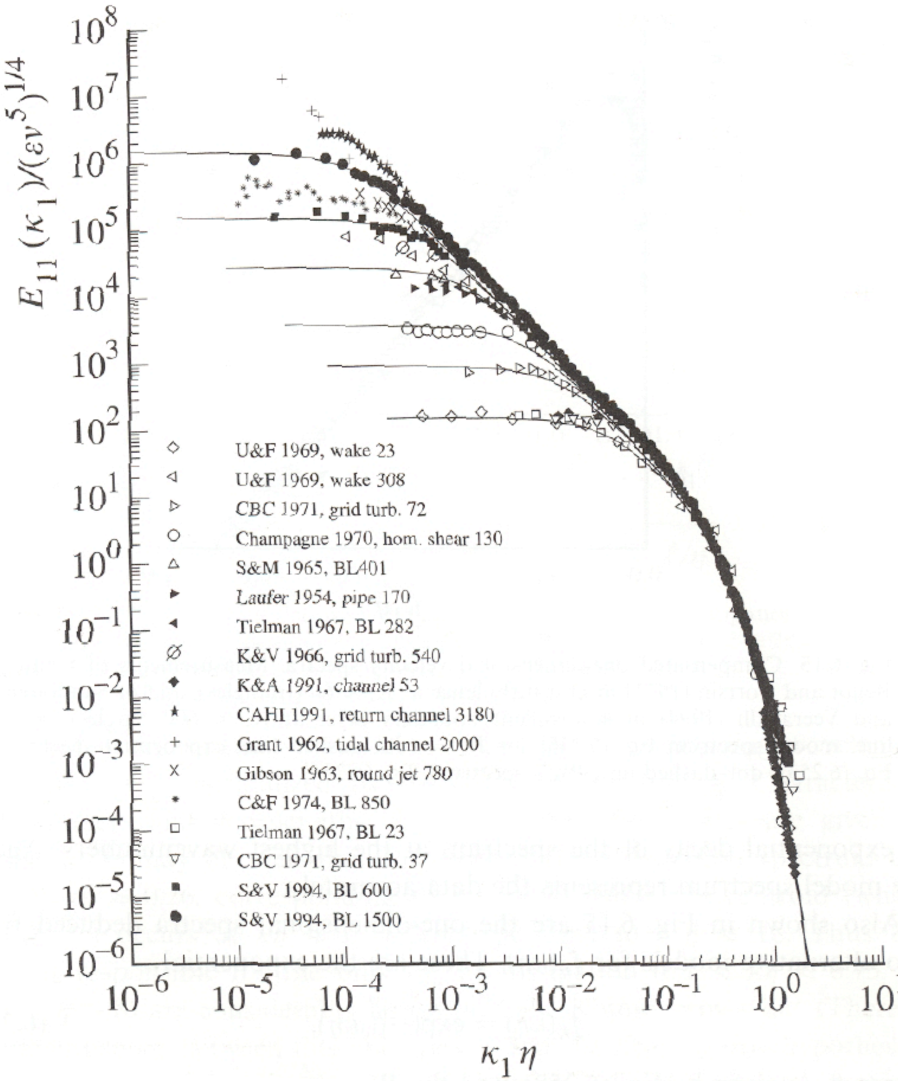
$$= \frac{\langle u_1^2\rangle}{\pi}\int_{-\infty}^{\infty} f(r)e^{-i\kappa_1 r_1}dr_1$$

Velocity variance

$$\langle u_1^2\rangle = R_{11}(0) = \int_0^{\infty} E_{11}(\kappa_1)d\kappa_1$$

Velocity Spectra

One dimensional spectrum



Kolmogorov Spectrum

- Consequences of Kolmogorov hypothesis for velocity spectrum tensor
- Hypothesis of local isotropy $(\kappa > \kappa_{EI})$

  $\Rightarrow$ velocity spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa})$ is given in terms of energy spectrum function $E(\kappa)$.
- First similarity hypothesis $(\kappa > \kappa_{EI})$

$$\Rightarrow \quad E(\kappa) = f(\kappa, \nu, \varepsilon) \qquad \text{Buck. } \Pi\text{-Theorem}, \quad NG = 4 - 2 = 2.$$

Kolmogorov Spectrum

$$\Rightarrow \quad E(\kappa) = f(\kappa, \nu, \varepsilon) \qquad \text{Buck. } \Pi\text{-Theorem}, \quad NG = 4 - 2 = 2.$$

- Non-dimensional wave number

$$\kappa\eta = \kappa \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

- Non-dimensional energy spectrum

$$\frac{E(\kappa)}{(\varepsilon\nu^5)^{1/4}} \qquad \text{or} \qquad \frac{E(\kappa)}{\varepsilon^{2/3}\kappa^{-5/3}}$$

Kolmogorov Spectrum

- For the energy spectrum follows

$$E(\kappa) = (\varepsilon\nu^5)^{1/4}\,\varphi(\kappa\eta) = u_\eta^2\,\eta\,\varphi(\kappa\eta)$$

$\varphi(\kappa\eta)$ is the Kolmogorov spectrum function.

Or alternatively

$$\frac{E(\kappa)}{\varepsilon^{2/3}\kappa^{-5/3}} = \psi\left(\kappa\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\right) = \psi(\kappa\eta)$$

$\psi(\kappa\eta)$ is the compensated Kolmogorov spectrum function.

$$E(\kappa) = \varepsilon^{2/3}\kappa^{-5/3}\psi(\kappa\eta)$$

Kolmogorov Spectrum

- Second similarity hypothesis $(\kappa_{EI} < \underbrace{\kappa < \kappa_{DI}}_{\kappa\eta \ll 1})$

$$\Rightarrow \quad \psi(\kappa\eta) \quad \text{constant}$$

$$\Rightarrow \quad E(\kappa) = C\varepsilon^{2/3}\,\kappa^{-5/3}; \quad C \approx 1.5, \quad \text{Kolmogorov constant}$$

Kolmogorov Spectrum

Model Spectrum

- From Kolmogorov hypothesis we found that for $\kappa < \kappa_{\mathrm{EI}}$ (length scales much smaller than the large scales of the turbulence), the velocity statistics should be isotropic and universal
- Hence, we can fit the universal part of the spectrum with an analytic function
- This model spectrum can then be used to look at some quantitative details of the energy cascade

Kolmogorov Spectrum

Model Spectrum

$$E(\kappa) = C\varepsilon^{2/3}\,\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta)$$

with

$$f_L(\kappa L) = \left(\frac{\kappa L}{\sqrt{(\kappa L)^2 + C_L}}\right)^{5/3+p_0}, \quad p_0 = 2, \quad C_L > 0$$

$$f_\eta(\kappa\eta) = \exp\left(-\beta\left([(\kappa\eta)^4 + C_\eta^4]^{1/4} - C_\eta\right)\right), \quad \beta = 5.2, \quad C_\eta > 0$$

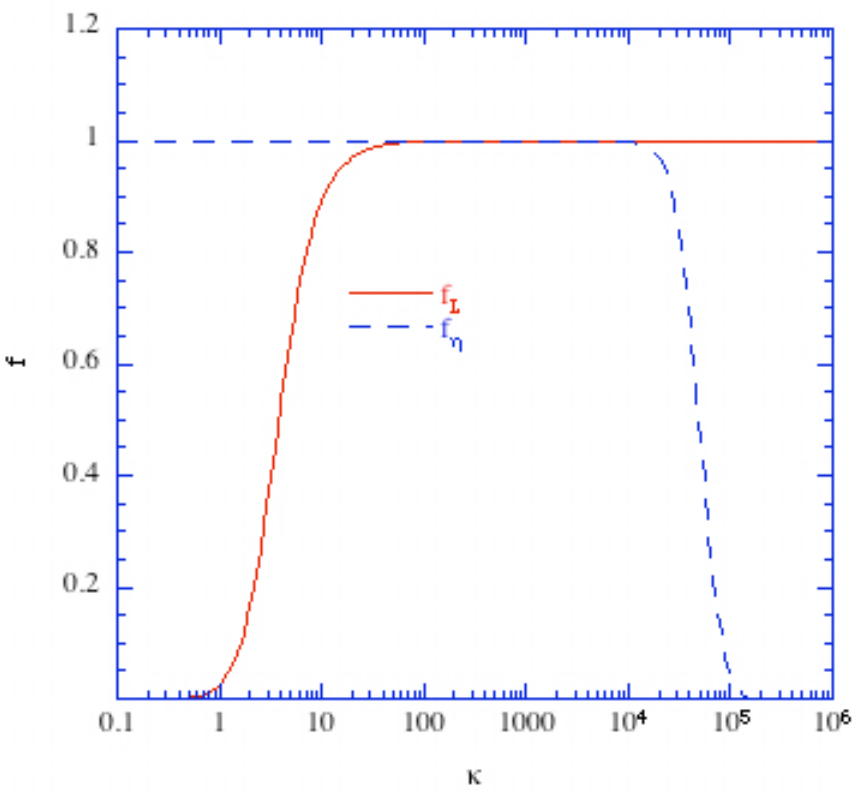
f_η decays faster than any power of $(\kappa\eta)$

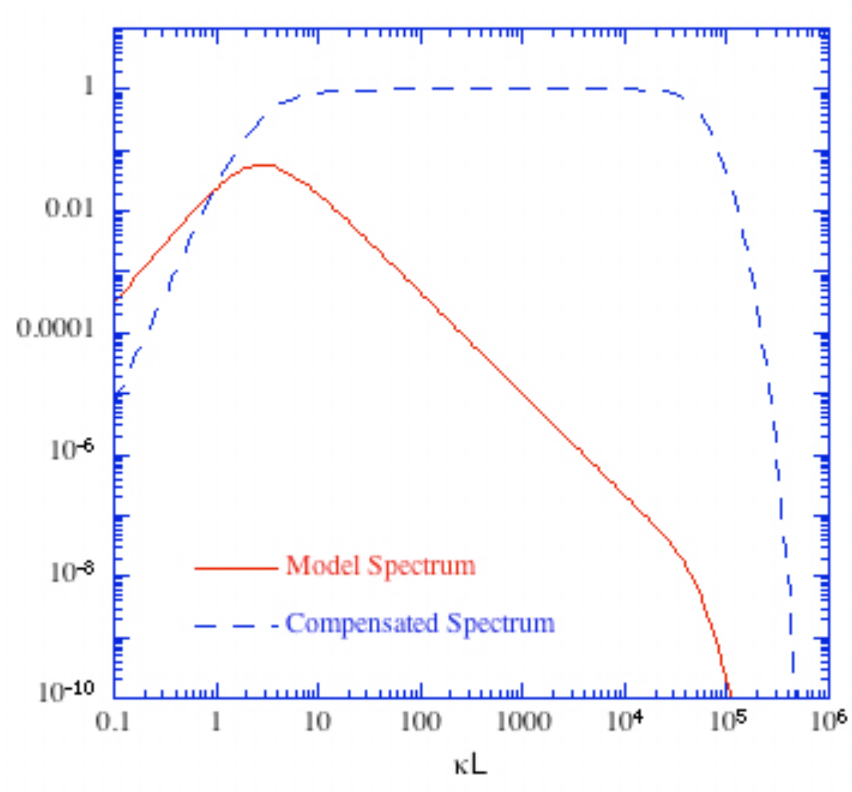
C_L and C_η are determined from

$$\left.\begin{array}{rcl} k & = & \int_0^\infty E(\kappa)d\kappa \\ \varepsilon & = & \int_0^\infty 2\nu\kappa^2 E(\kappa)d\kappa \end{array}\right\} \Rightarrow C_L = 6.7, \quad C_\eta = 0.4$$

Kolmogorov Spectrum

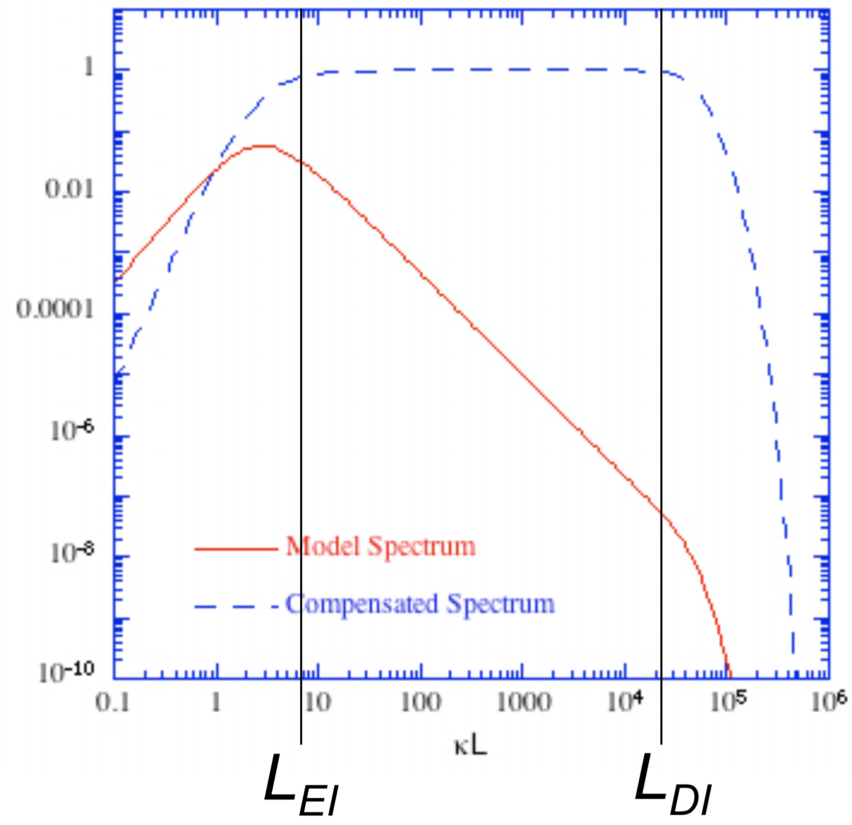
$$E(\kappa) = C\varepsilon^{2/3}\,\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta)$$





Kolmogorov Spectrum

- Determine L_{EI}
- Determine L_{DI}



Kolmogorov Spectrum

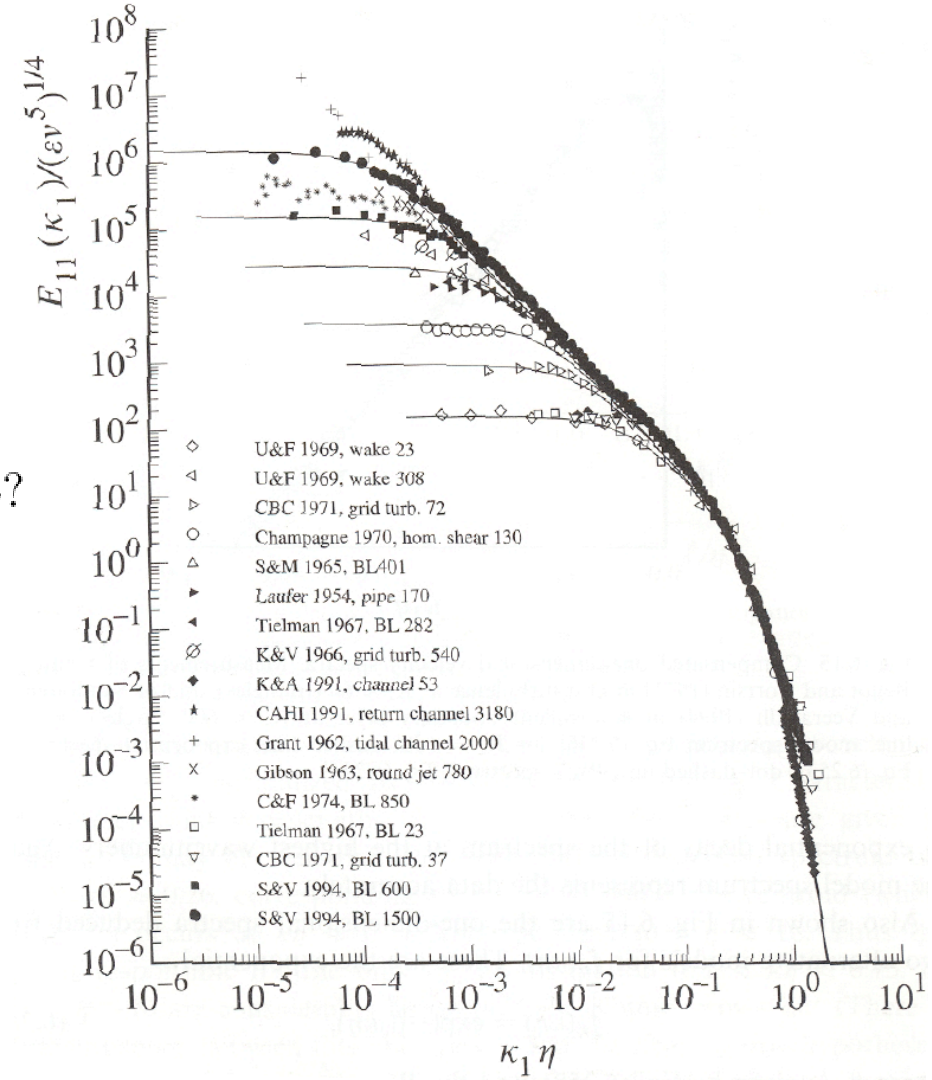
Discussion of the Velocity Spectra

Dissipation Range

One dimensional longitudinal velocity spectra

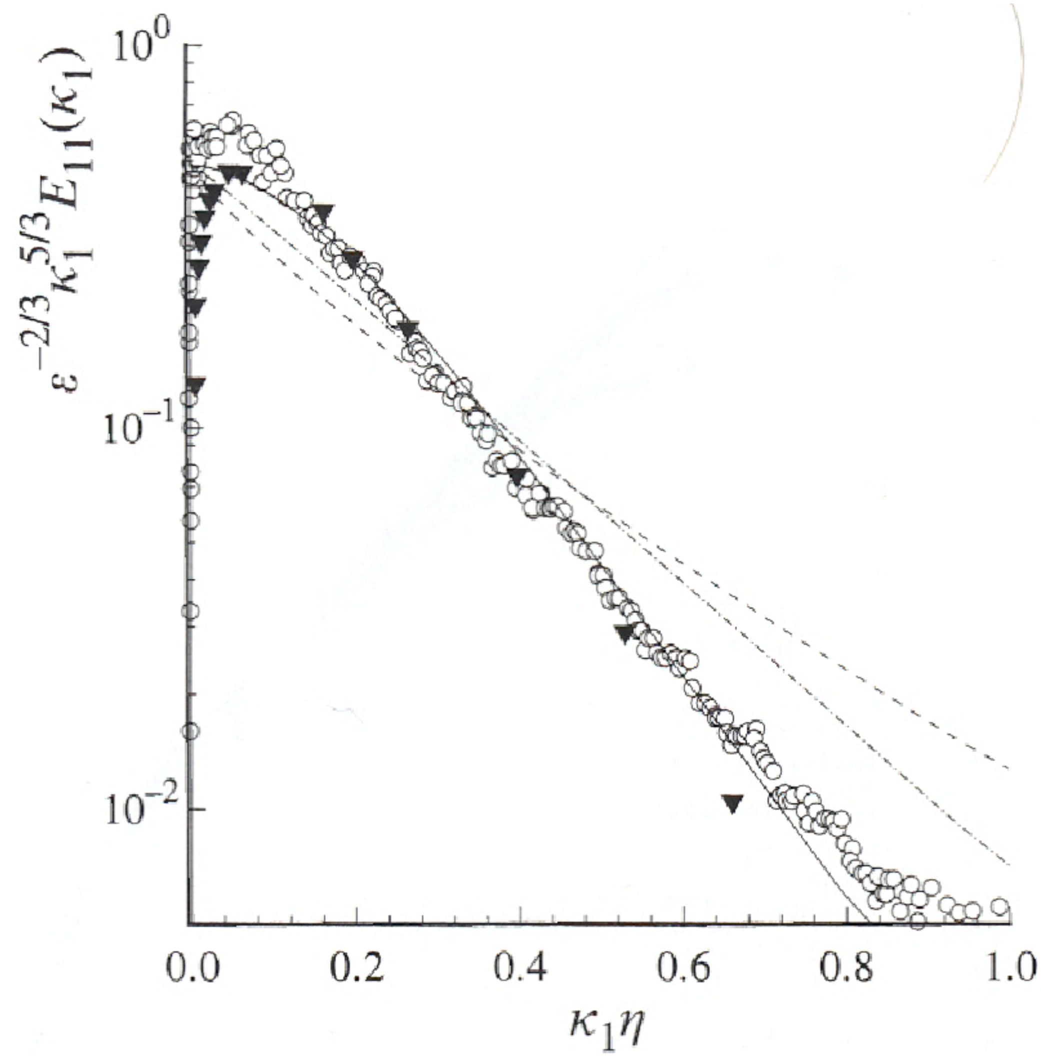
$$E_{11}(\kappa_1) = (\nu^5 \varepsilon)^{1/4} \varphi(\kappa\eta)$$

- Power-law behavior for what value of κ, L?



Kolmogorov Spectrum

Compensated 1-D velocity spectrum



- Exponential decay for $\kappa_1\eta > 0.1$
- Model spectrum in good agreement

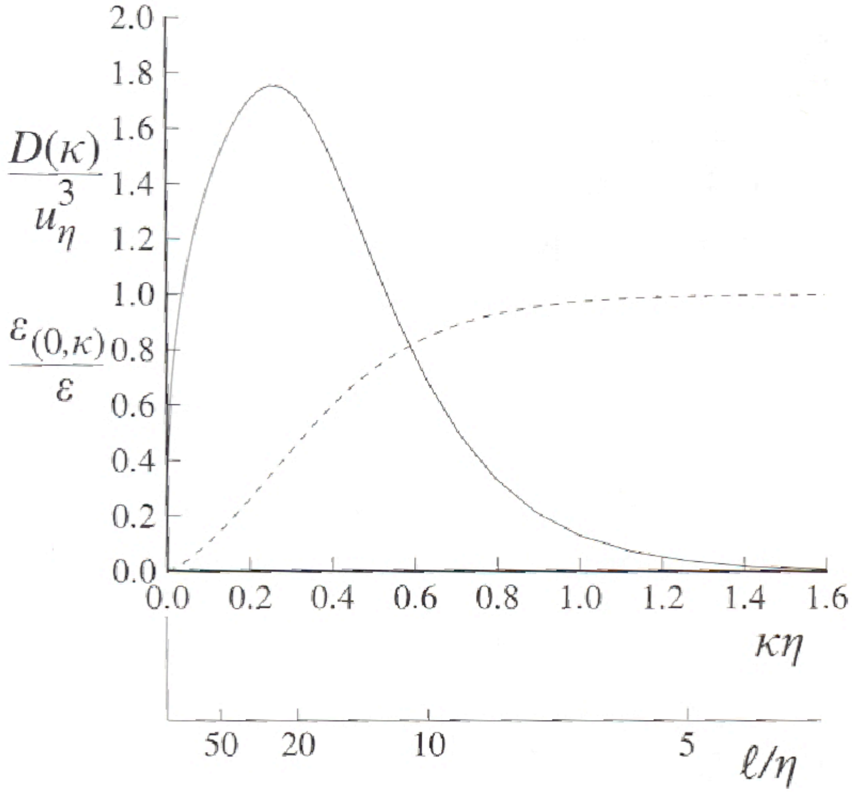
Kolmogorov Spectrum

Dissipation spectrum $D(\kappa)$

and cumulative dissipation $\varepsilon_{0,\kappa}$

$$\varepsilon_{0,\kappa} = \int_0^{\kappa} D(\kappa')d\kappa' \quad \text{with} \quad D(\kappa) = 2\nu\kappa^2 E(\kappa)$$

- $\varepsilon_{0,\kappa} \approx 0.1$ at $\kappa\eta \approx 0.1 \Rightarrow L_{DI} \approx 60\eta$; $\varepsilon_{0,\kappa} \approx 0.9$ at $\kappa\eta \approx 0.7 \Rightarrow L_s \approx 8\eta$

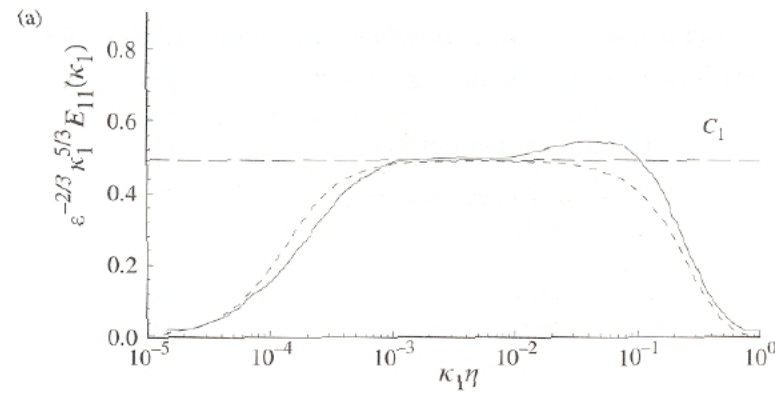


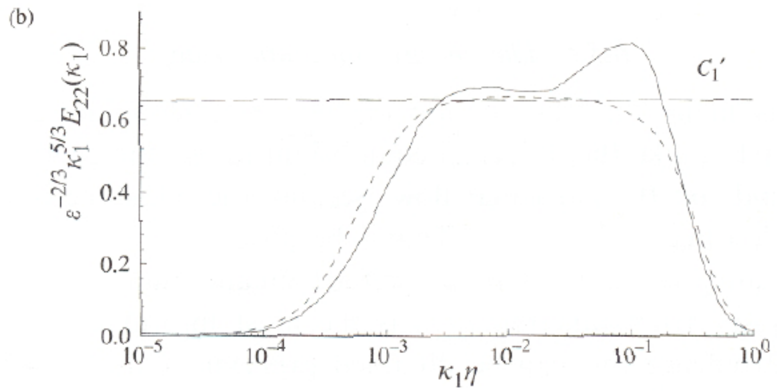
Kolmogorov Spectrum

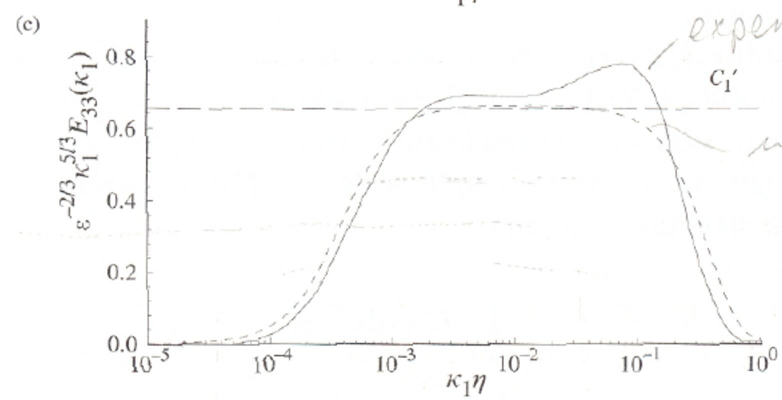
Inertial Subrange

Compensated spectrum

$$\frac{E_{11}(\kappa_1)}{\varepsilon^{2/3}\kappa^{-5/3}} \quad \text{as function of} \quad \kappa_1\eta$$



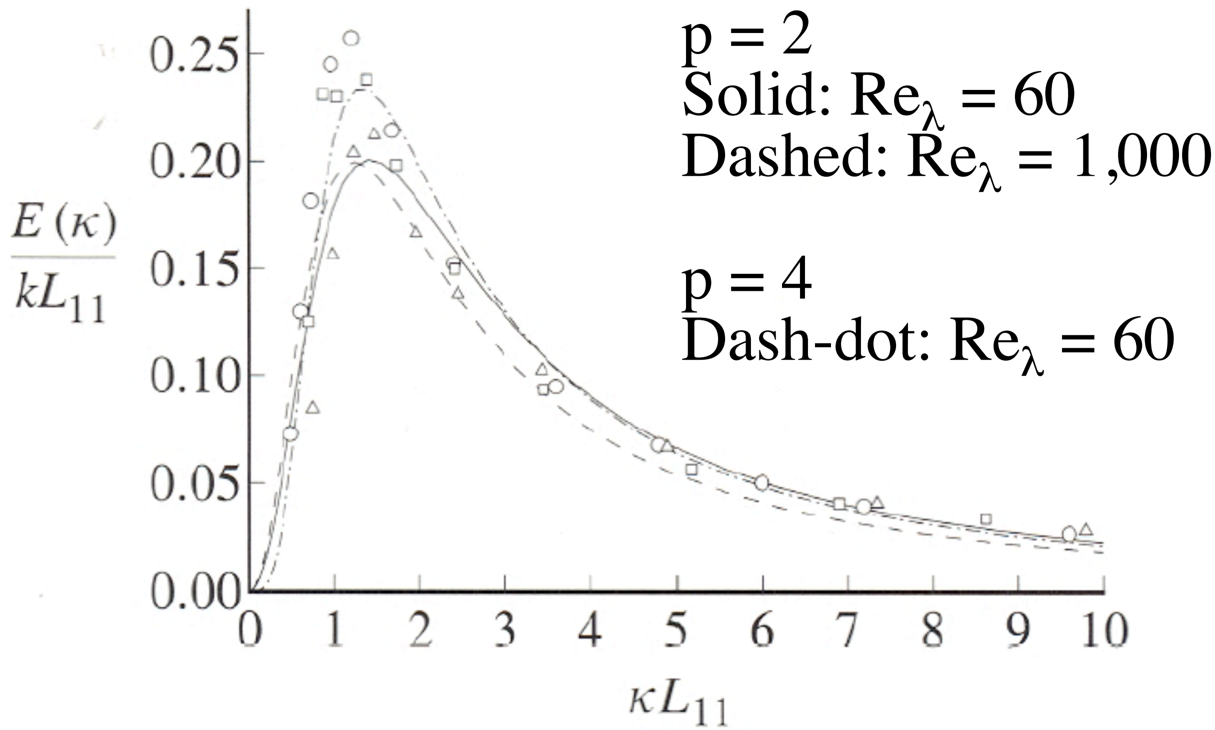




- $E_{11,22,33}$ from experiment show inertial range over two orders of magnitude, (BL: $Re_\lambda = 1450$).
- E_{22} and E_{33} are similar $\Rightarrow$ local isotropy.
- $E_{11} = \frac{3}{4}E_{22}$ in inertial subrange $\Rightarrow$ result from local isotropy.

Kolmogorov Spectrum

Energy containing range



- Model spectrum agrees well with experiments.

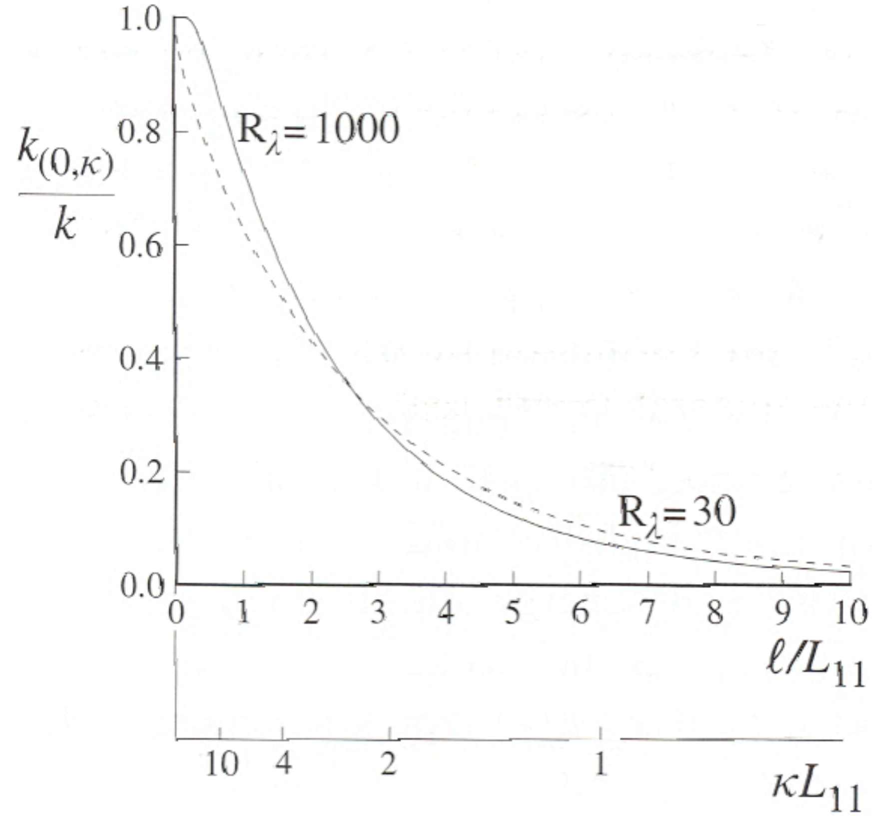
Kolmogorov Spectrum

Cumulative turbulent kinetic energy

$$k_{0,\kappa} = \int_0^\kappa E(\kappa')d\kappa'$$

- 80% of k between $1/6L_{11}$ and $6L_{11}$

$$\Rightarrow \qquad L_{EI} = \frac{1}{6}L_{11}$$



Kolmogorov Spectrum

Energy Spectrum Balance

Balance equation for energy spectrum function

$$\frac{\partial}{\partial t}E(\kappa,t) = P_k(\kappa,t) - \frac{\partial}{\partial \kappa}T_\kappa(\kappa,t) - 2\nu\kappa^2 E(\kappa,t)$$

- P_k depends on mean velocity gradients $\frac{\partial \langle U \rangle_i}{\partial x_j}$ and anisotropic part of the spectrum tensor

  $\Rightarrow \quad P_k$ only at $\kappa < \kappa_{EI}$
- $\frac{\partial T(\kappa)}{\partial \kappa}$ is spectral energy transfer rate
- Dissipation spectrum $D(\kappa) = 2\nu\kappa^2 E(\kappa)$ only at large κ

Kolmogorov Spectrum

Balance equation for energy spectrum function

$$\frac{\partial}{\partial t}E(\kappa,t) = P_k(\kappa,t) - \frac{\partial}{\partial \kappa}T_\kappa(\kappa,t) - 2\nu\kappa^2 E(\kappa,t)$$

Integrate from 0 to κ_{EI}

$$\frac{dk}{dt} = P - T_{EI}$$

Integrate from k_{EI} to k_{DI}

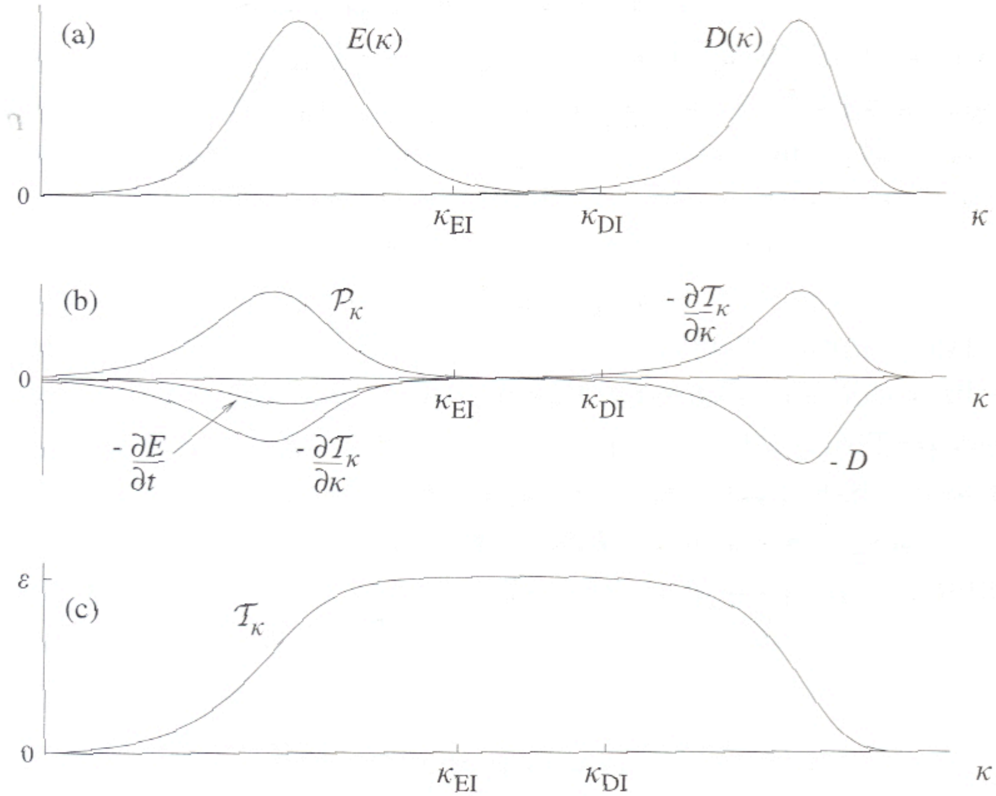
$$0 = T_{DI} - T_{EI}$$

Integrate from k_{DI} to ∞

$$0 = T_{DI} - \varepsilon$$

Sum them up

$$\frac{dk}{dt} = P - \varepsilon$$



Channel Flow

- In technical systems, flows are often confined by walls
- Moreover, turbulence production requires velocity gradients, which are often generated by wall boundary layers and no-slip condition
- Examples:
  - External aerodynamics, such as aircraft, golf ball
  - Internal flows, such as turbine, compressor, pipes and pipelines, rivers, straw
- Because of the importance of the no-slip condition, this must be properly incorporated in a theory for wall bounded flows

Channel Flow

- Turbulent structures have to decrease if the wall is approached
- Questions we want to study:
  - How is the turbulence in external turbulent boundary layers, pipes, and channels different/similar from that of free shear flows?
  - What is the structure of the flow in close vicinity to the wall?
- Canonical configurations:
  - External turbulent boundary layers, turbulent flow in pipes and channels

Channel Flow

Definition

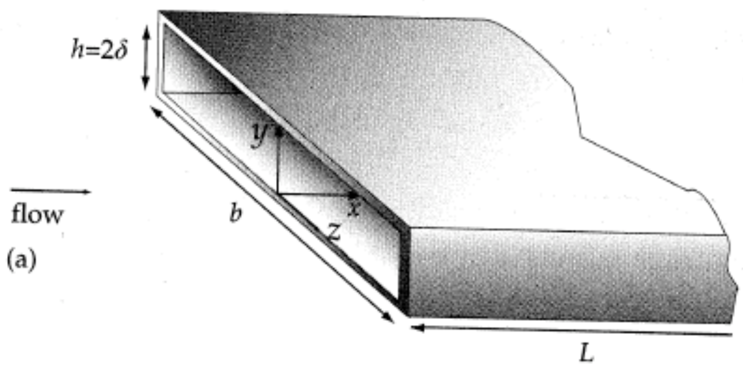


Fig. 7.1 from Pope's Turbulent Flows book

Assumptions:

$$L \gg \delta$$

$$b \gg \delta$$

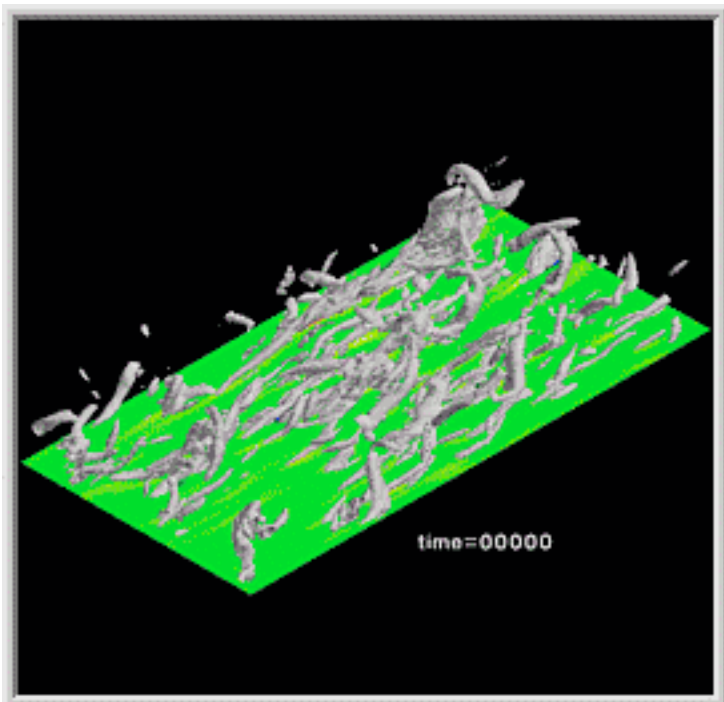
$\Longrightarrow$ Turbulence statistics are homogeneous in x and z

Reynolds numbers: $\mathrm{Re} = \dfrac{2\delta\overline{U}}{\nu}$; $\overline{U}$ is bulk velocity

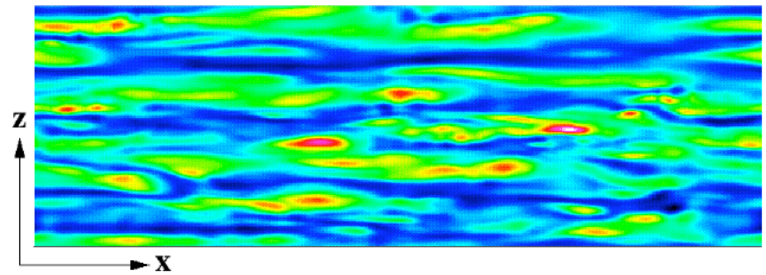
$\mathrm{Re}_0 = \dfrac{\delta U_0}{\nu}$; $U_0 = \langle U \rangle_{y=\delta}$ is centerline velocity

Fully developed for $\mathrm{Re} > 1800$

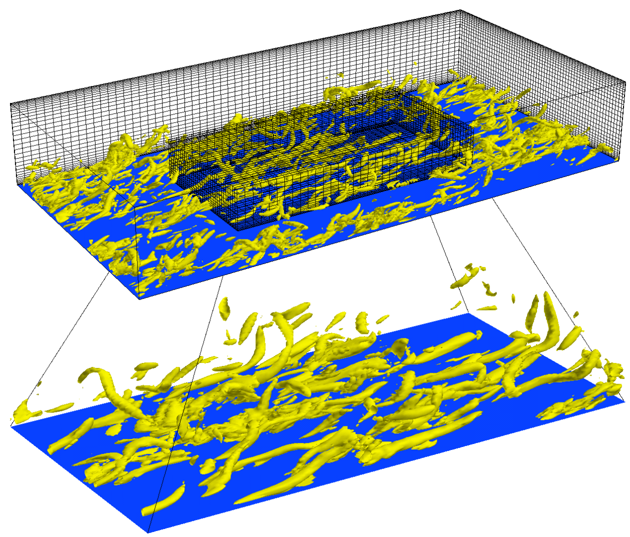
Channel Flow



DNS of turbulent channel flow by Bewley, Hammond, and Moin (1987)

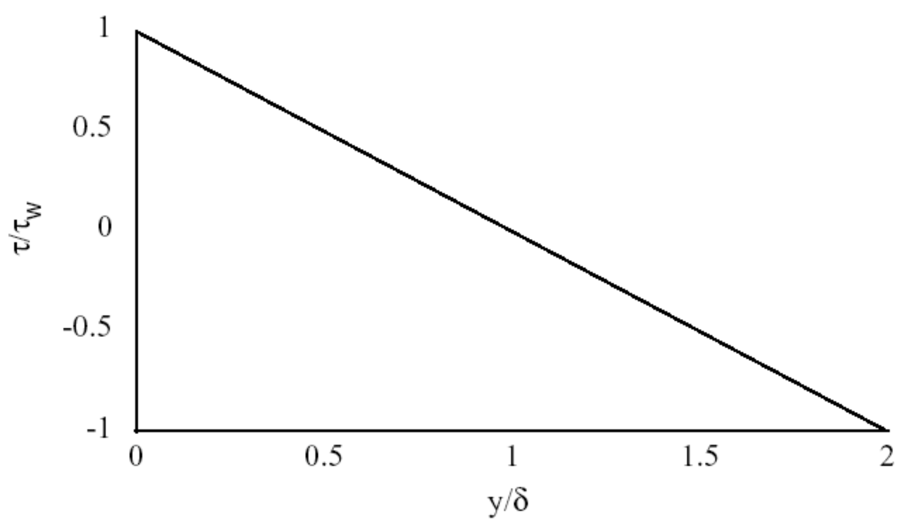


Skin friction from DNS of turbulent channel flow by Kim, Moin, and Moser (1987)



Channel Flow

$$\left.\begin{array}{rcl} \dfrac{\partial \tau}{\partial y} & \neq & f(x) \\ \dfrac{\partial p_w}{\partial x} & \neq & f(y) \end{array}\right\} \Longrightarrow \frac{\partial \tau}{\partial y} = \frac{\partial p_w}{\partial x} = \text{const}$$



Shear stress τ/τ_w as function of y/δ

Skin friction coefficients

$$c_f \equiv \frac{\tau_w}{1/2\rho U_0^2}$$

$$C_f \equiv \frac{\tau_w}{1/2\rho \overline{U}^2}$$

Channel Flow

Near Wall Shear Stesses

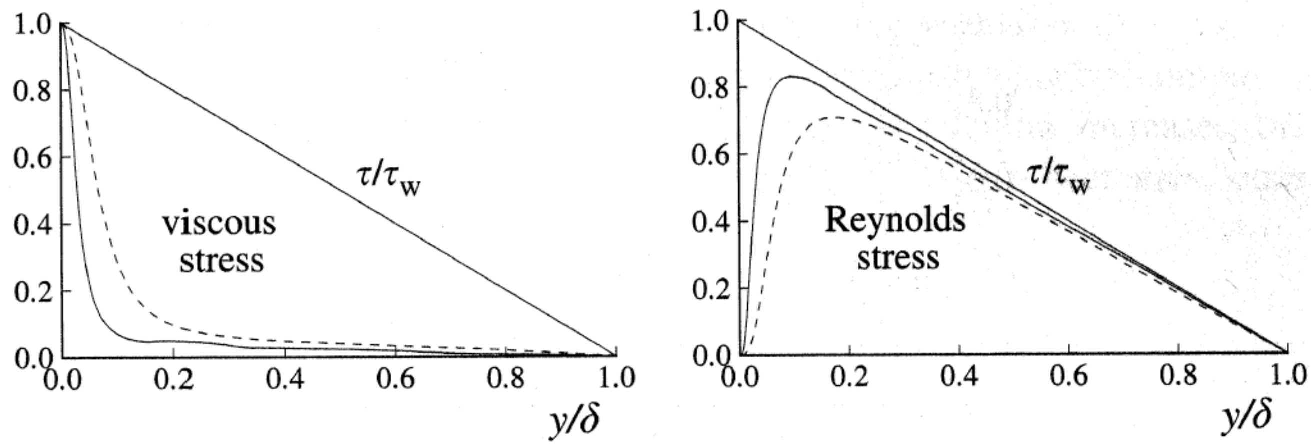


Figure 7.3 from Pope's Turbulent Flows book: Reynolds stress, viscous stress, and total stress (DNS of Kim, Moin, Moser (1987)

τ_w given only by viscous stress

$$\tau_w = \rho\nu \left. \frac{\partial \langle U \rangle}{\partial y} \right|_{y=0}$$

Channel Flow

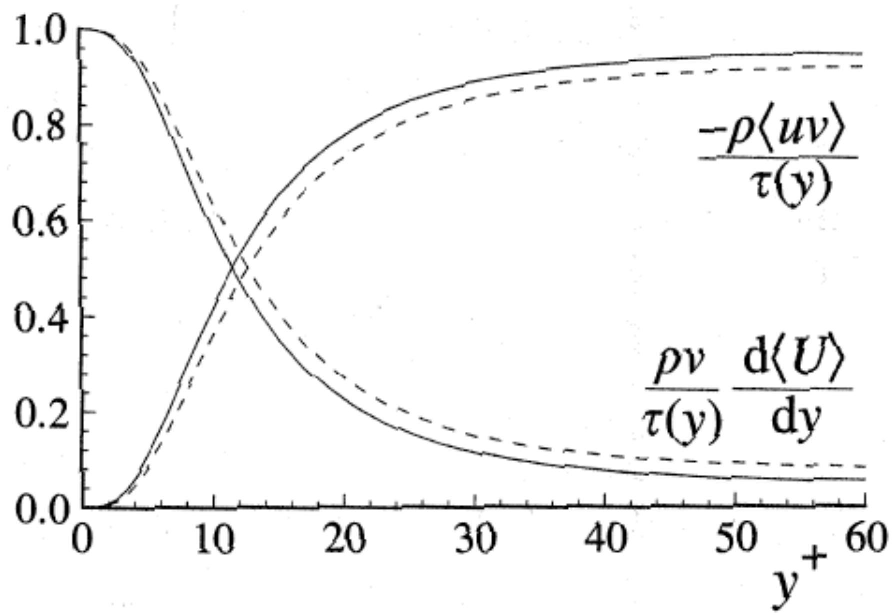


Figure 7.4 from Pope's Turbulent Flows book: Contribution of Reynolds and viscous stresses to total stress (DNS of Kim, Moin, Moser (1987))

- Outer layer: $y^+ > 50$
- Viscous wall region: $y^+ < 50$
- Viscous sublayer: $y^+ < 5$

Channel Flow

Viscous Sublayer

$$\Longrightarrow \qquad u^+ = y^+$$

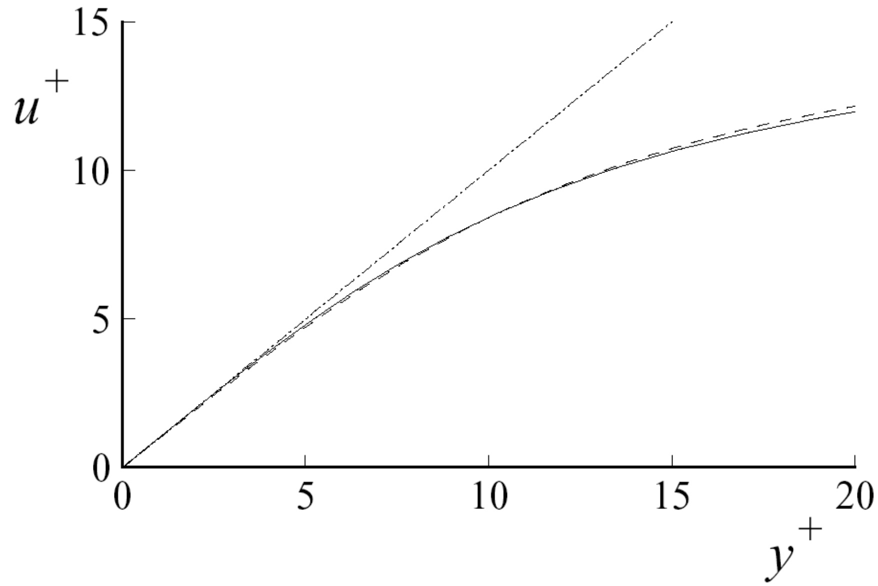
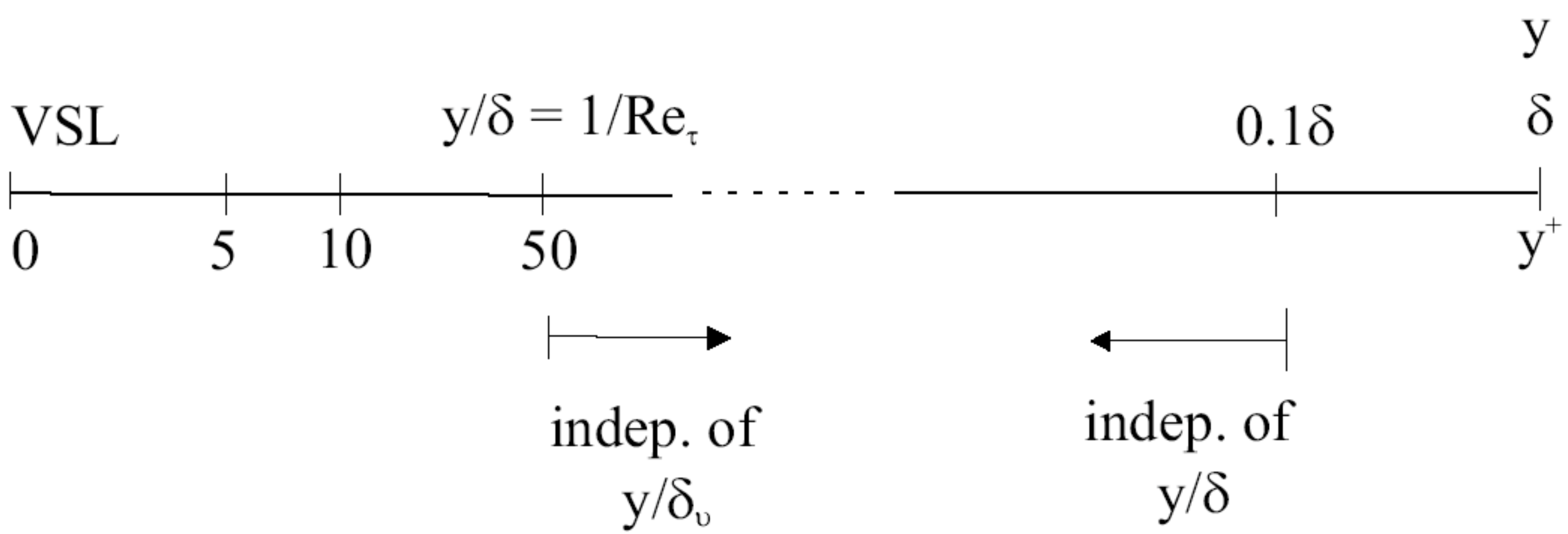


Figure 7.5: Near-wall profiles of mean velocity from the DNS data of Kim *et al.*: dashed line, Re = 5,600; solid line, Re = 13,750; dot-dashed line, $u^+ = y^+$.

Channel Flow



Channel Flow

Experimental and DNS Data

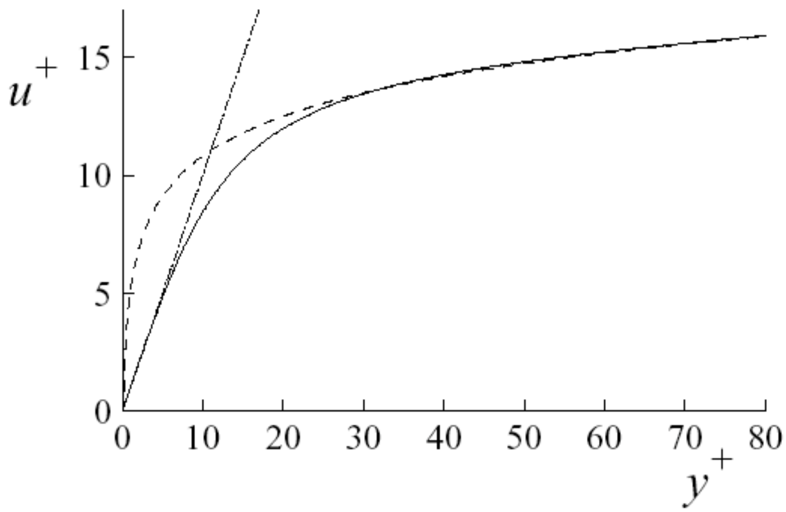


Figure 7.6: Near-wall profiles of mean velocity: solid line, DNS data of Kim *et al.*: Re = 13,750; dot-dashed line, $u^+ = y^+$; dashed line, the log law, Eqs. (7.43)–(7.44).

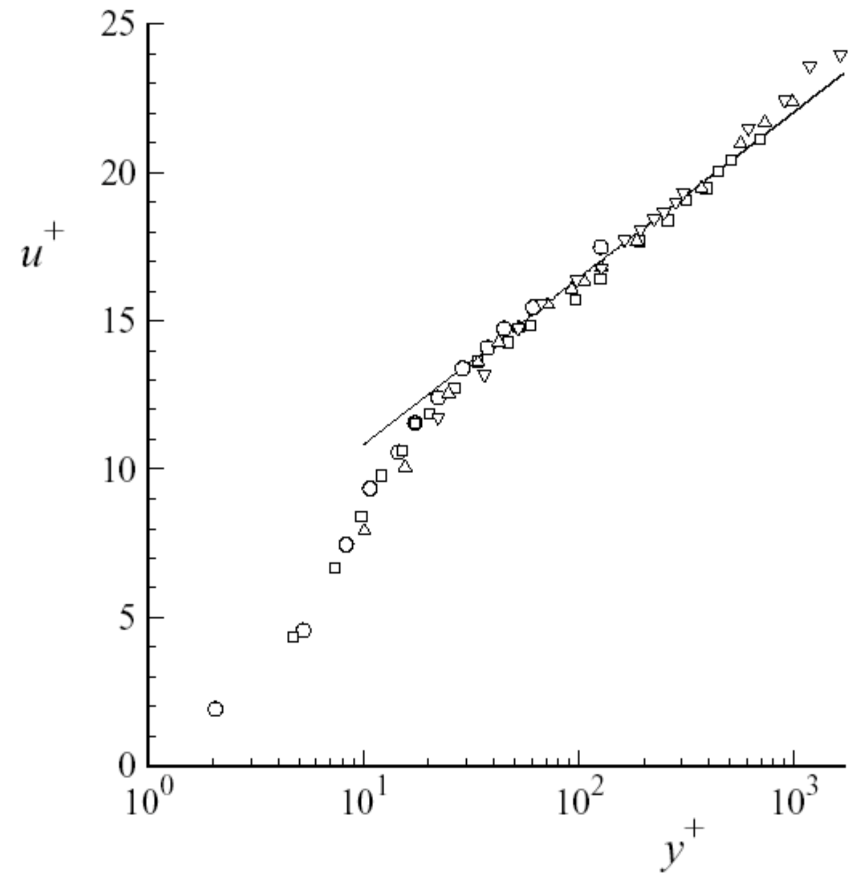


Figure 7.7: Mean velocity profiles in fully-developed turbulent channel flow measured by Wei and Willmarth (1989): ○, $Re_0 = 2,970$; □, $Re_0 = 14,914$; △, $Re_0 = 22,776$; ▽, $Re_0 = 39,582$; line, the log law, Eqs. (7.43)–(7.44).

Channel Flow

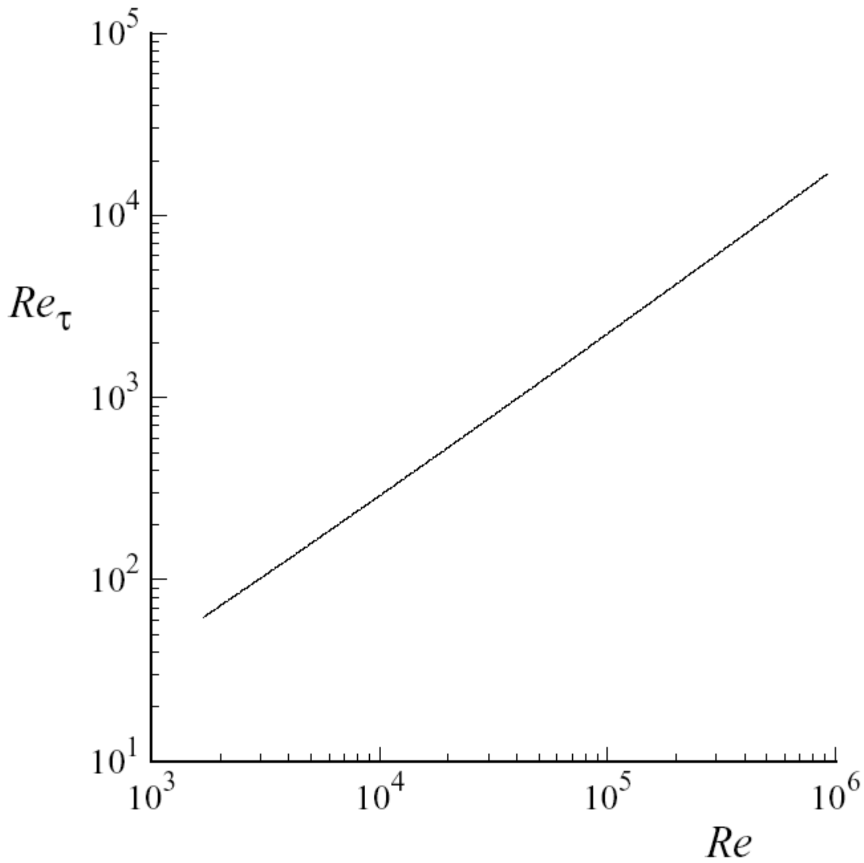


Figure 7.11: Outer-to-inner lengthscale ratio $\delta/\delta_\nu = \mathrm{Re}_\tau$ for turbulent channel flow as a function of Reynolds number (obtained from Eq. 7.55).

Channel Flow

Turbulent Kinetic Energy Budget

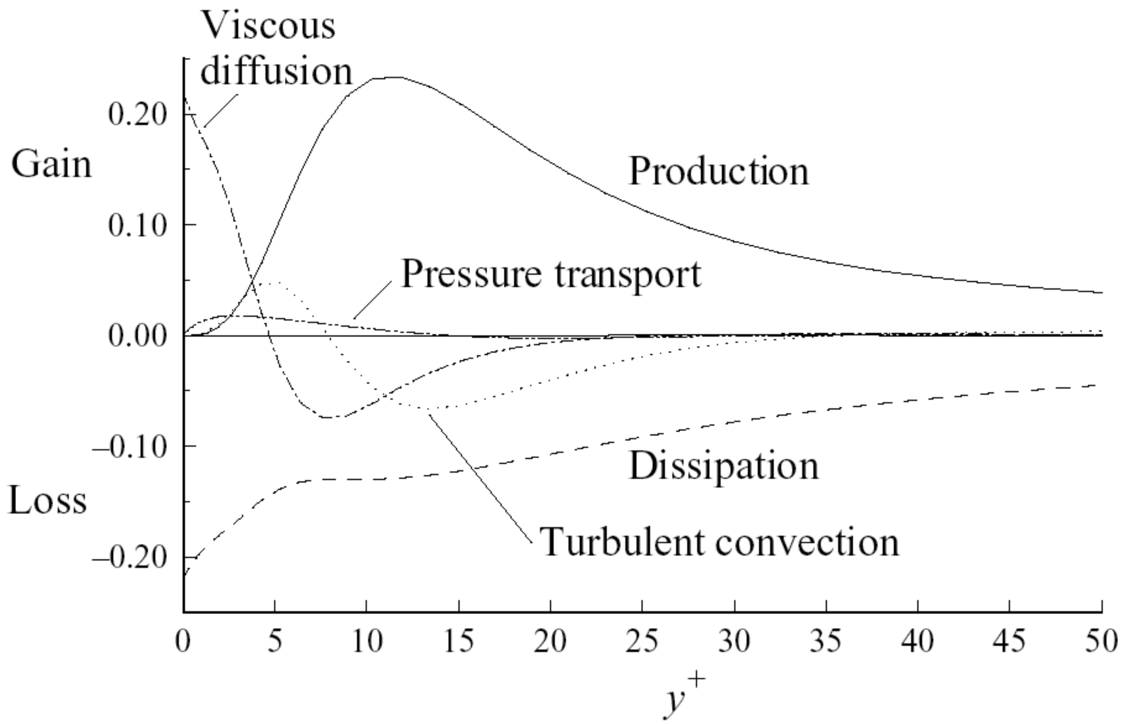
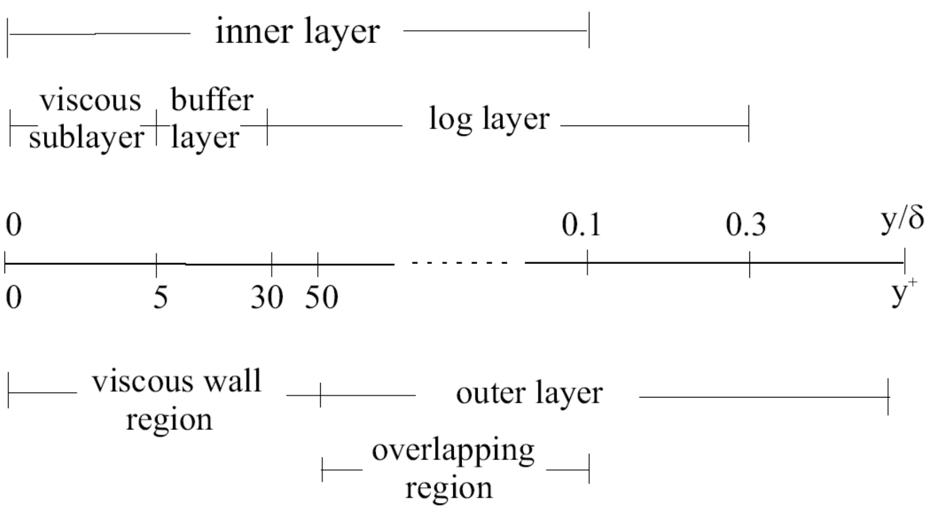
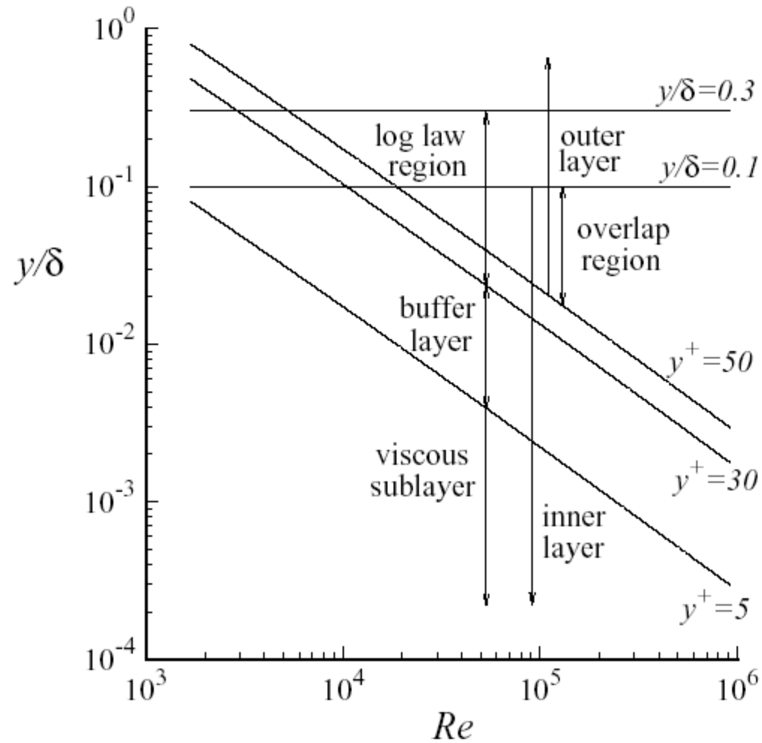


Figure 7.18: Turbulent kinetic energy budget in the viscous wall region of channel flow: terms in Eq. (7.64) normalized by viscous scales. From the DNS data of Kim *et al.* (1987) Re = 13, 750.

Channel Flow

Law of the Wall: Summary





Boundary Layer

Definition

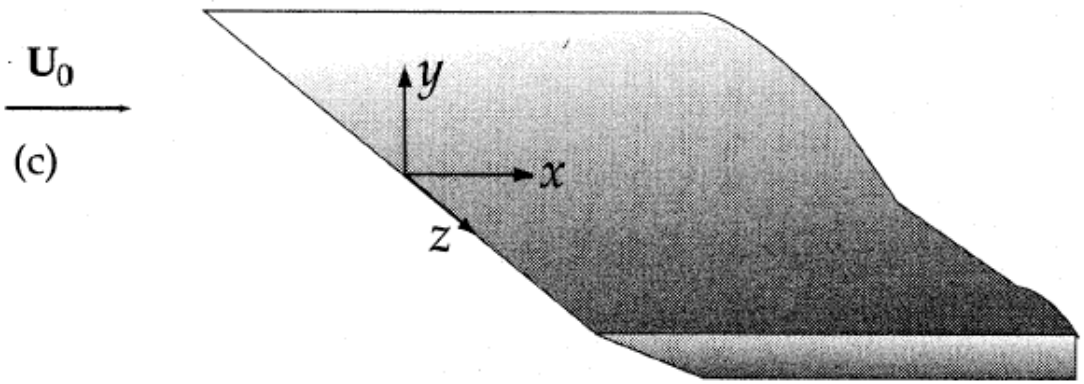


Fig. 7.1 from Pope's Turbulent Flows book

Assumptions: Statistics independent of z: $\langle W \rangle = 0$

Boundary Layer

Differences to Turbulent Channel Flow

- Boundary layer develops in flow direction, $\delta = \delta(x)$
- τ_w not known a priori
- Intermittency

But: Inner layer $(y/\delta(x) < 0.1)$ very similar to channel flow

Boundary Layer

Bernoulli for free stream

$$p_0(x) + \frac{1}{2}\rho U_0^2(x) = \text{const}$$

$$\frac{\partial p_0(x)}{\partial x} + \rho U_0(x)\frac{\partial U_0(x)}{\partial x} = 0$$

Boundary layer thickness $\delta(x)$ from

$$\left\langle U \right\rangle (x,y)\big|_{y=\delta} = 0.99 U_0(x)$$

Boundary Layer

Reynolds numbers

$$\mathrm{Re}_x \equiv \frac{U_0 x}{\nu}, \qquad \mathrm{Re}_\delta \equiv \frac{U_0 \delta}{\nu}, \qquad \mathrm{Re}_{\delta^*} \equiv \frac{U_0 \delta^*}{\nu}, \qquad \mathrm{Re}_\Theta \equiv \frac{U_0 \Theta}{\nu}$$

Critical Reynolds number

$$\mathrm{Re}_{x,\mathrm{crit}} \approx 10^6$$

Boundary Layer

Integration of momentum equation leads to von Karman integral momentum equation

(Derivation for $\frac{\partial \langle p_0 \rangle}{\partial x} = 0$)

$$\frac{\partial \langle U \rangle^2}{\partial x} + \frac{\partial \langle U \rangle \langle V \rangle}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\int_0^\infty \frac{\partial \langle U \rangle^2}{\partial x} dy + \int_0^\infty \frac{\partial \langle U \rangle \langle V \rangle}{\partial y} dy = \frac{1}{\rho}\int_0^\infty \frac{\partial \tau}{\partial y} dy$$

$$-\int_0^\infty \frac{\partial}{\partial x}\left[\langle U \rangle (U_0 - \langle U \rangle)\right] dy + \int_0^\infty U_0 \underbrace{\frac{\partial \langle U \rangle}{\partial x}}_{\text{Continuity: } -\frac{\partial \langle V \rangle}{\partial y}} dy + \underbrace{\left[\langle U \rangle \langle V \rangle\right]_0^\infty}_{=0} = \frac{1}{\rho}\underbrace{\left[\tau\right]_0^\infty}_{=-\tau_w}$$

$$-U_0^2 \frac{\partial \Theta}{\partial x} - U_0 \underbrace{\int_0^\infty \frac{\partial \langle V \rangle}{\partial y} dy}_{\left[\langle V \rangle\right]_0^\infty = 0} = -\frac{\tau_w}{\rho}$$

$$\Longrightarrow \qquad \tau_w = \rho U_0^2 \frac{\partial \Theta}{\partial x}$$

Boundary Layer

Mean Velocity Profiles

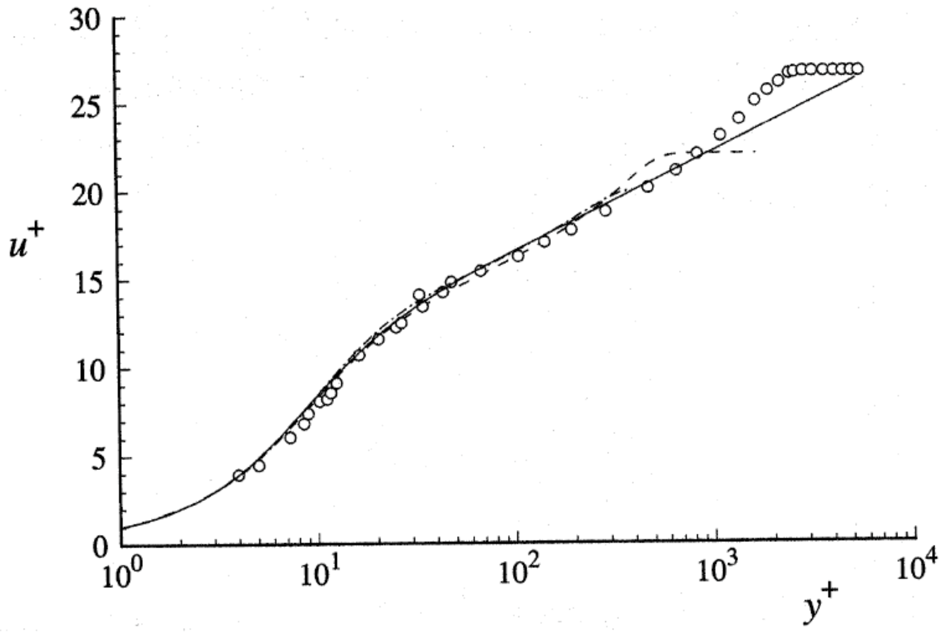


Fig. 7.27 from Pope's Turbulent Flows book

Boundary Layer

Log layer:

$\Longrightarrow$ Can be solved with $l_m^+ = \kappa y^+$

Viscous sublayer:

$\Longrightarrow$ $\langle uv \rangle = -\nu_t \frac{\partial \langle U \rangle}{\partial y} = -(\kappa y^+)^2 \frac{\partial \langle U \rangle}{\partial y} \sim y^{+2}$

$\Longrightarrow$ incorrect y^+ dependence (should be y^{+3})

$\Longrightarrow$ Van Driest damping function assures proper transition to viscous sublayer

$$l_m^+ = \kappa y^+ \left[1 - \exp\left(-\frac{y^+}{A^+} \right) \right] \quad \text{with} \quad A^+ = 26$$

Boundary Layer

Velocity Defect Law

Velocity in whole boundary layer well predicted by

$$\frac{\langle U\rangle}{u_\tau} = \underbrace{f_w\left(\frac{y}{\delta_\nu}\right)}_{\text{Law of the wall}} + \underbrace{\frac{\Pi}{\kappa}w\left(\frac{y}{\delta}\right)}_{\text{Law of the wake}}$$

Wake function $w(y/\delta)$ universal, defined to yield

$$w(0)=1 \qquad \text{and} \qquad w(1)=2$$

Wake strength parameter Π flow dependent

Approximate f_w from log law

$$\frac{\langle U\rangle}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{y}{\delta_\nu}\right) + B + \frac{\Pi}{\kappa}w\left(\frac{y}{\delta}\right)$$

Boundary Layer

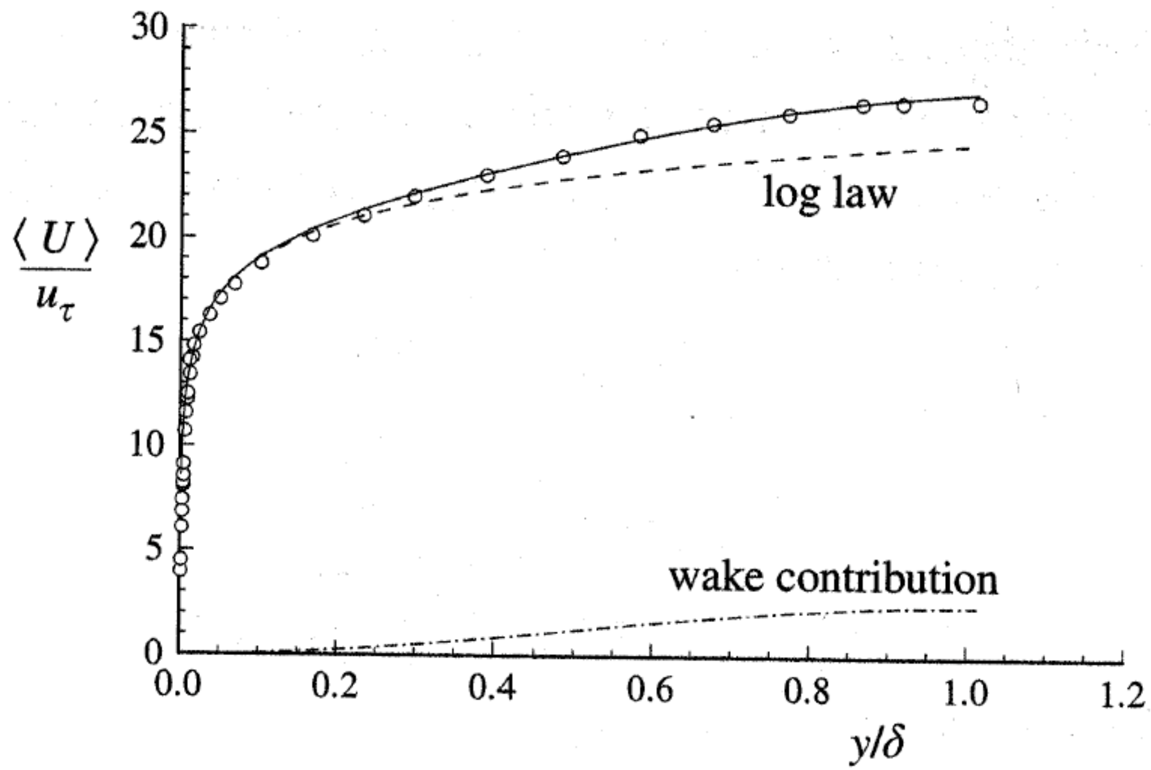


Fig. 7.28 from Pope's Turbulent Flows book

Boundary Layer

For $y = \delta$

$$\frac{U_0}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{\delta}{\delta_\nu}\right) + B + \frac{2\Pi}{\kappa}$$

$$= \frac{1}{\kappa}\ln\left(\mathrm{Re}_\delta \frac{u_\tau}{U_0}\right) + B + \frac{2\Pi}{\kappa}$$

$\Longrightarrow$ Can be solved for u_τ/U_0

$\Longrightarrow$ Skin friction coefficient

$$c_f = \frac{\tau_w}{1/2\rho U_0^2} = 2\left(\frac{u_\tau}{U_0}\right)^2$$

Boundary Layer

Eddy Viscosity in Defect Layer

Eddy viscosity definition

$$\nu_t = \frac{\tau}{\rho \partial \langle U \rangle / \partial y}$$

Eddy viscosity model

$$\nu_t = l_m^2 \frac{\partial \langle U \rangle}{\partial y}$$

Defect layer: τ smaller and $\frac{\partial \langle U \rangle}{\partial y}$ larger as given by log law

$\Longrightarrow$ ν_t smaller as given by log law

$\Longrightarrow$ l_m has to be smaller than κy

$\Longrightarrow$ $l_m = \kappa y$ incorrect in defect layer, has to be adjusted

For example

$$l_m = \min(\kappa y, 0.09\delta)$$

Boundary Layer

Reynolds Stress Equation

Reynolds stress components of comparable magnitude

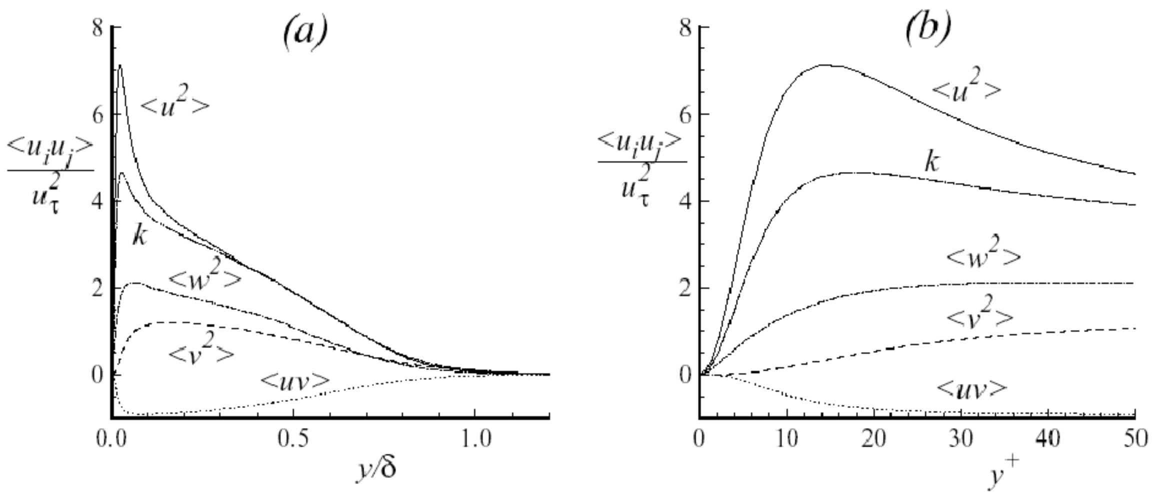


Figure 7.33: Profiles of Reynolds stresses and kinetic energy normalized by the friction velocity in a turbulent boundary layer at $\mathrm{Re}_\theta = 1,410$: (a) across the boundary layer (b) in the viscous near-wall region. From the DNS data of Spalart (1988).

Boundary Layer

Turbulent kinetic energy budget

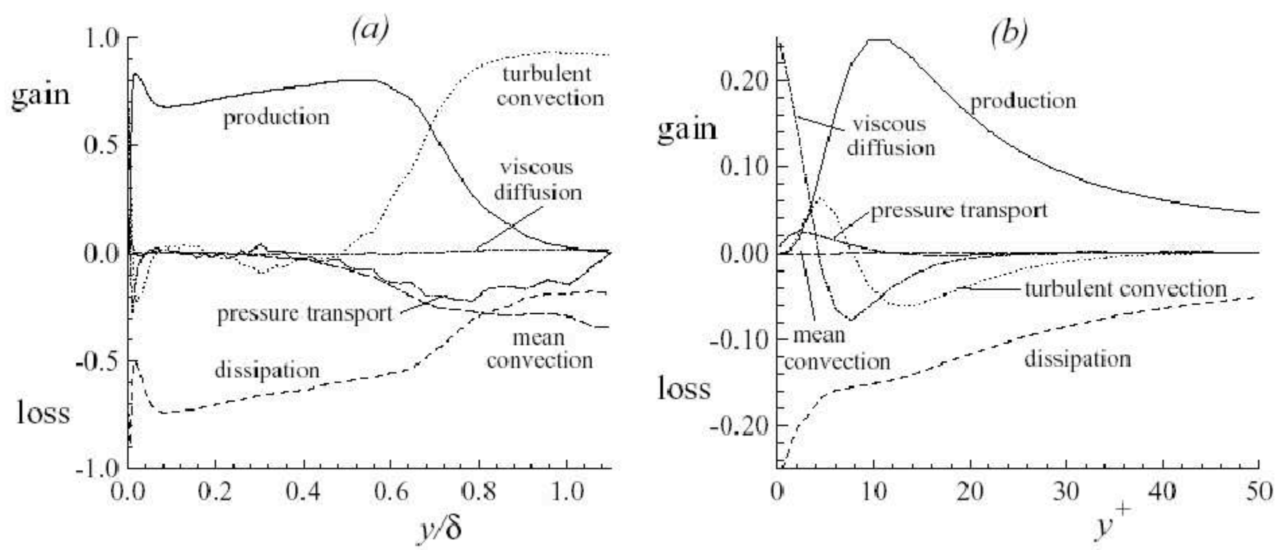


Figure 7.34: Turbulent kinetic energy budget in a turbulent boundary layer at $Re_\theta = 1,410$: terms in Eq. (7.177) (a) normalized as a function of y so that the sum of the squares of the terms is unity (b) normalized by the viscous scales. From the DNS data of Spalart (1988).

Boundary Layer

Reynolds stress budgets:

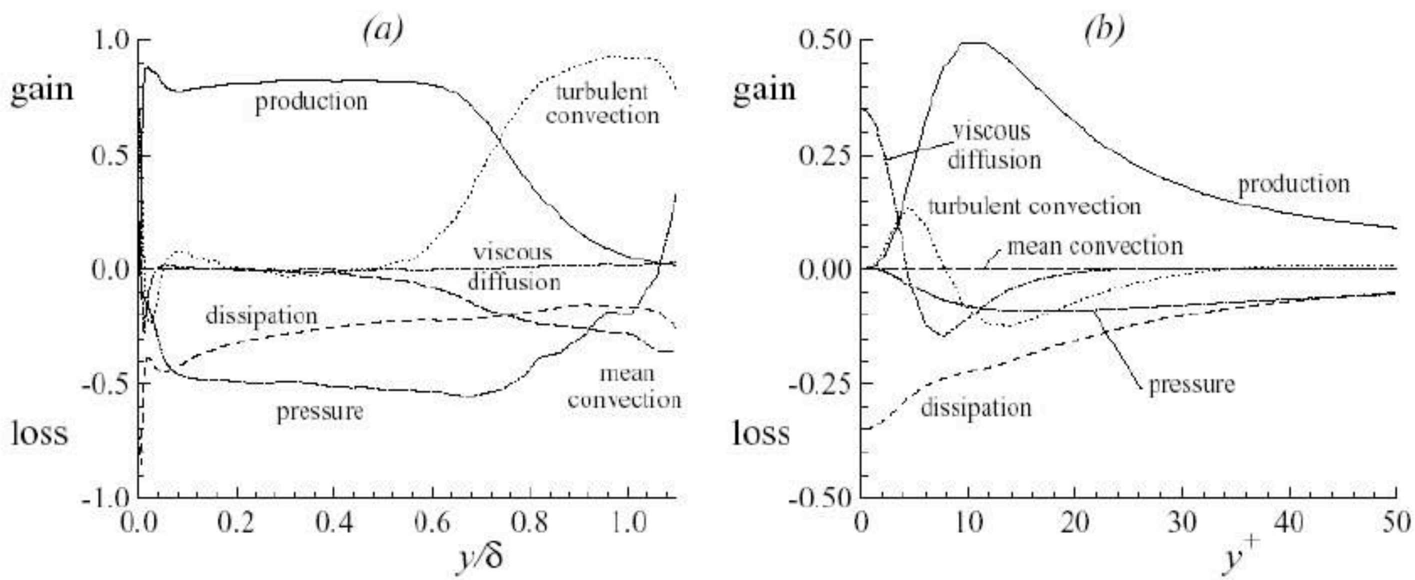


Figure 7.35: Budget of $\langle u^2 \rangle$ in a turbulent boundary layer: conditions and normalization are the same as in Fig. 7.34.

Boundary Layer

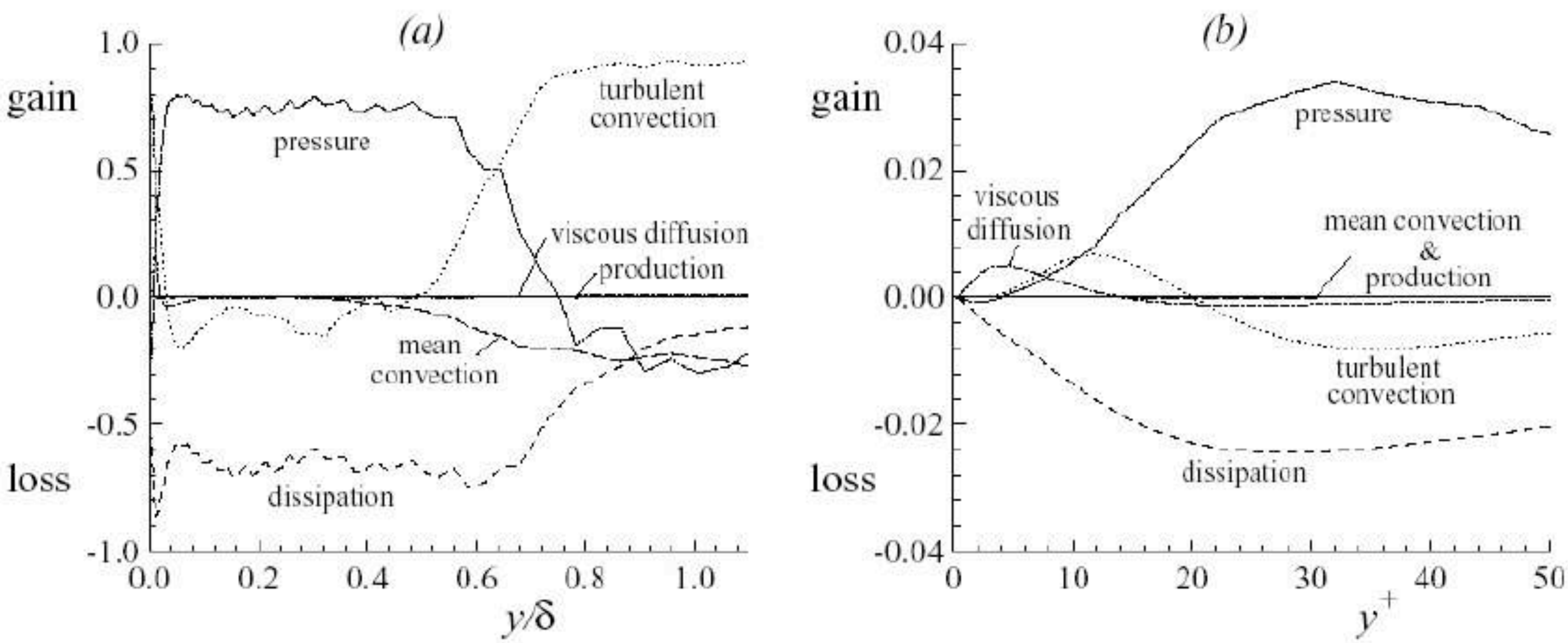


Figure 7.36: Budget of $\langle v^2 \rangle$ in a turbulent boundary layer: conditions and normalization are the same as in Fig. 7.34.

Boundary Layer

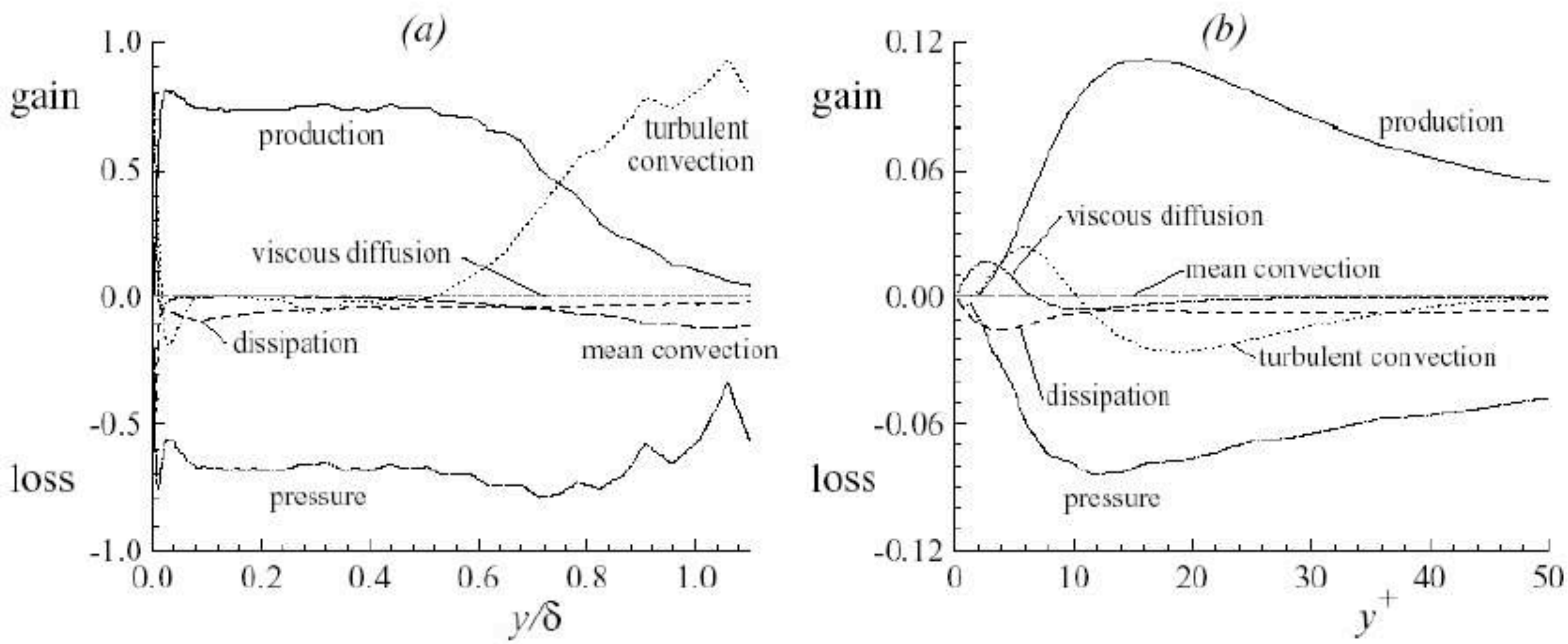
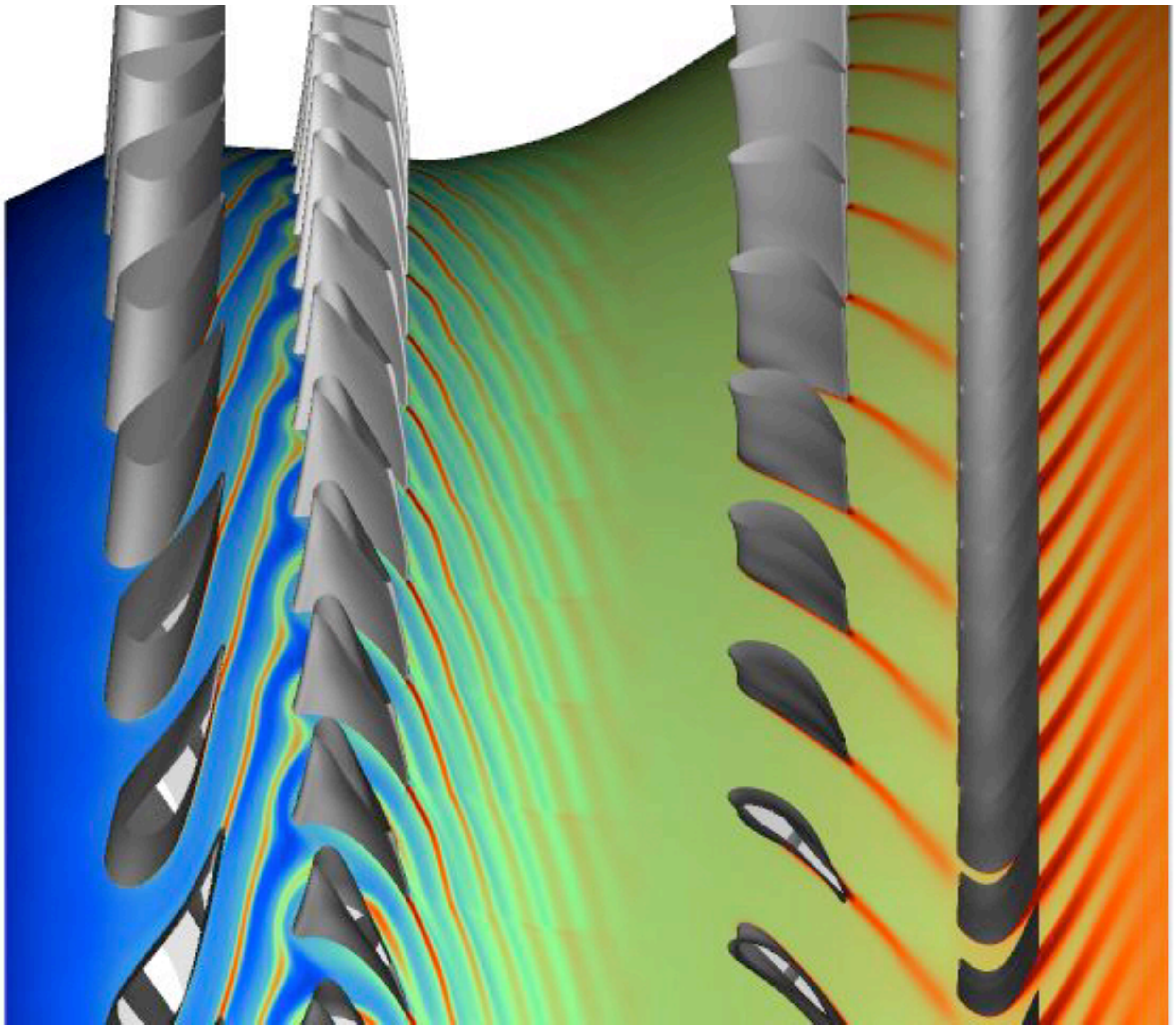
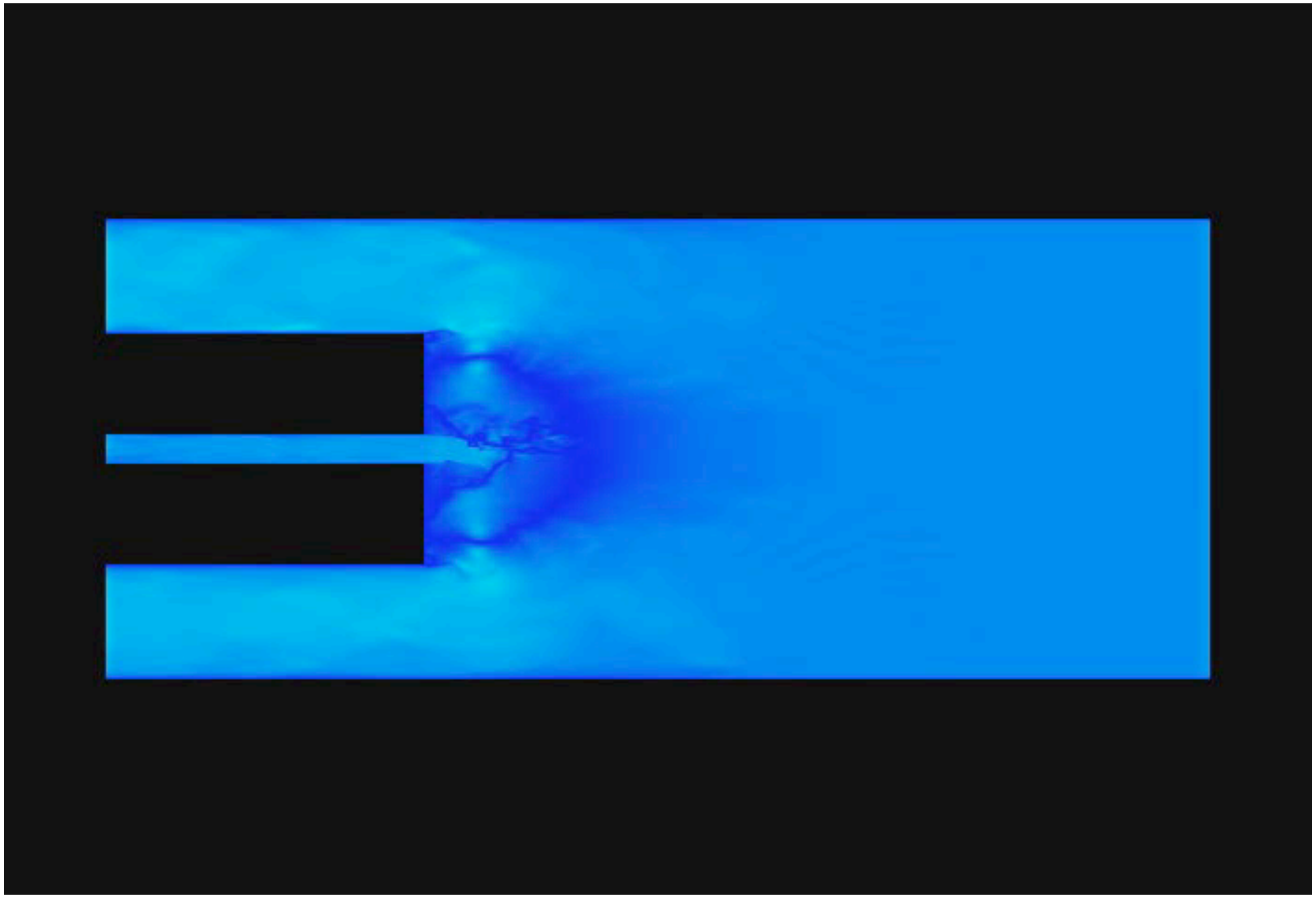
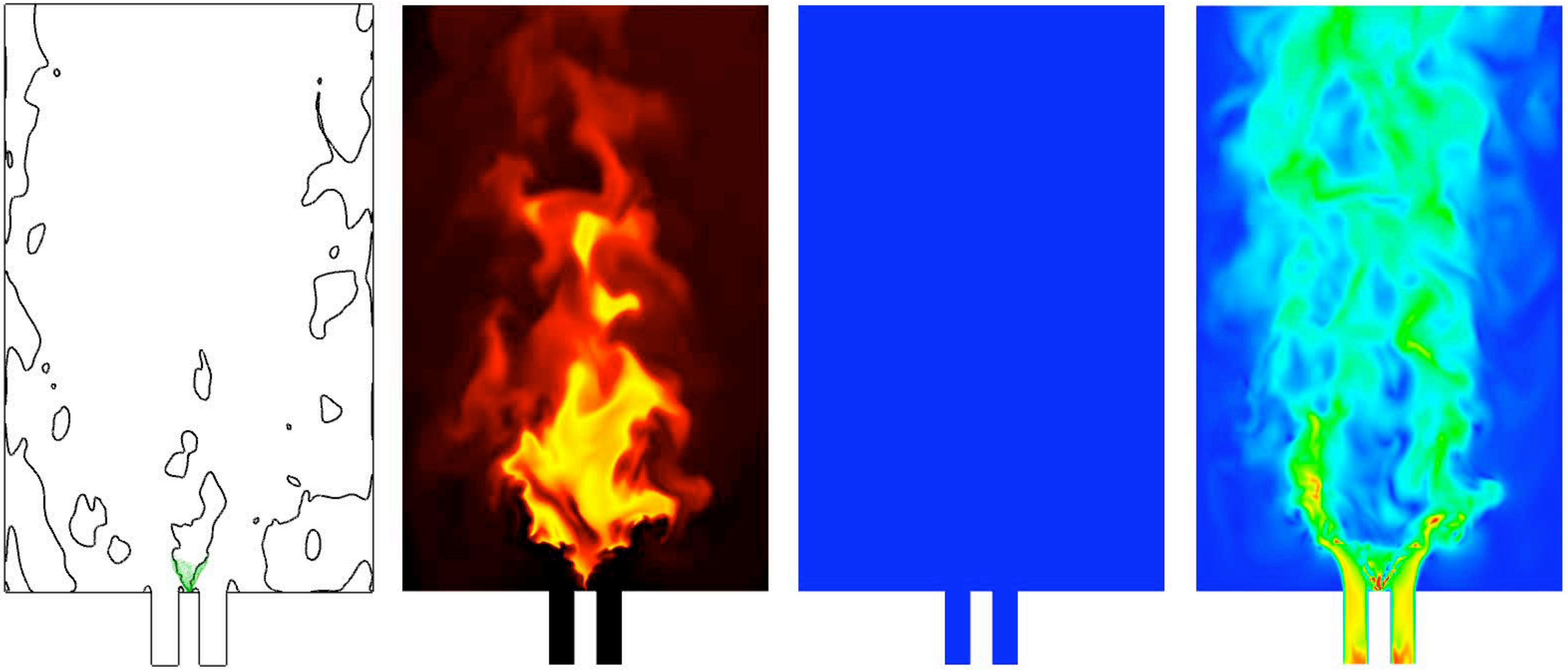


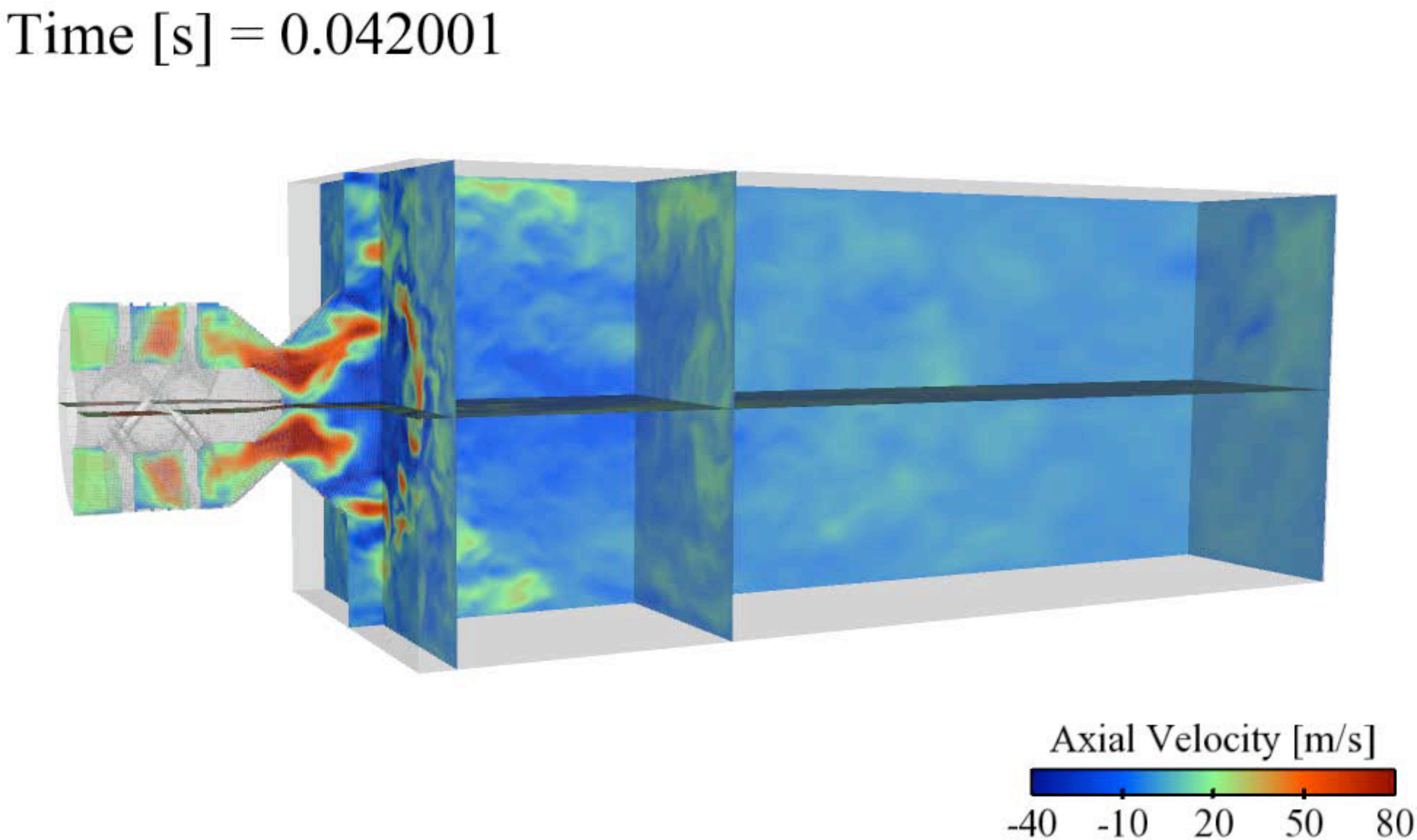
Figure 7.38: Budget of $-\langle uv \rangle$ in a turbulent boundary layer: conditions and normalization are the same as in Fig. 7.34.

Turbulence Modeling

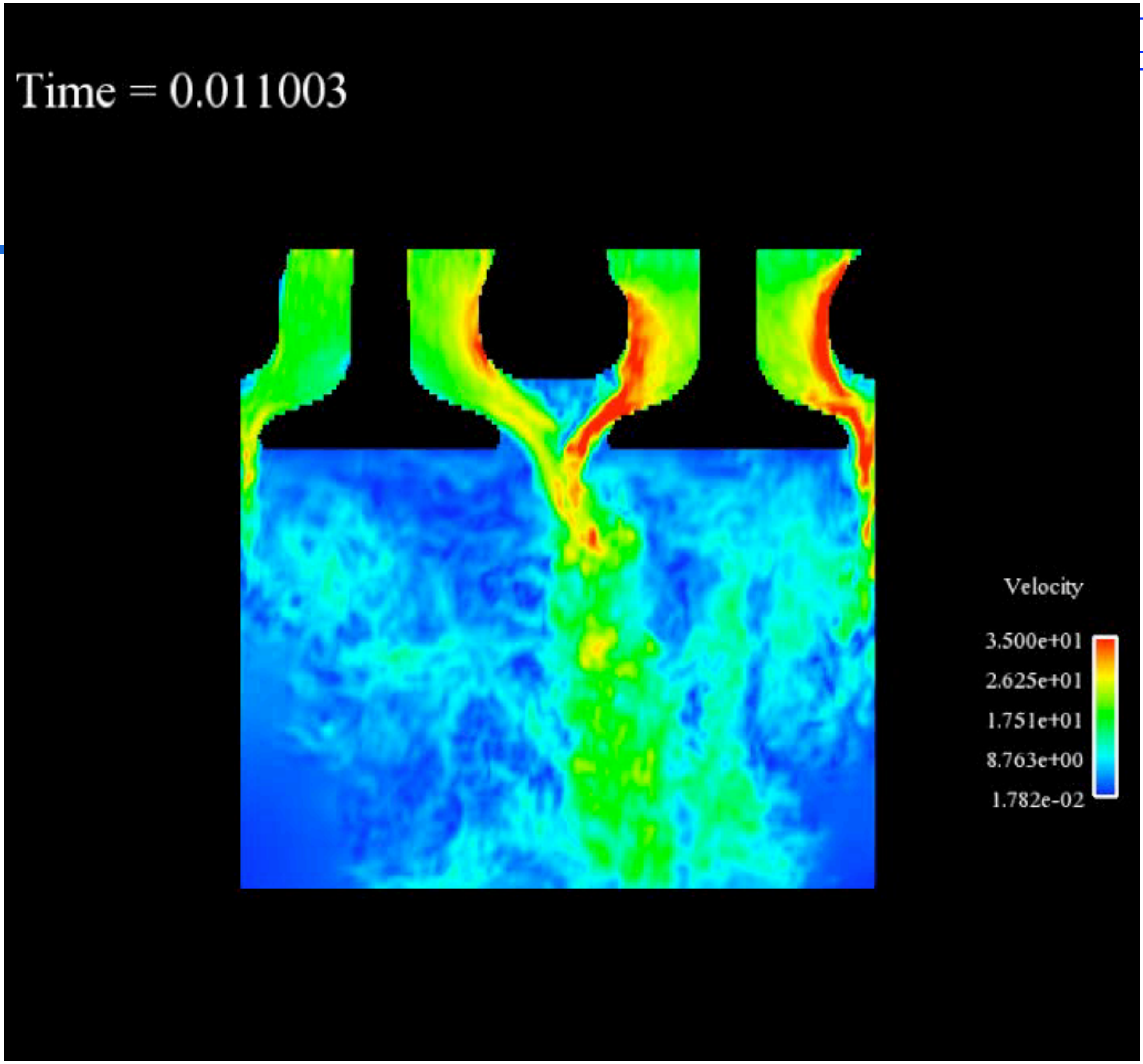


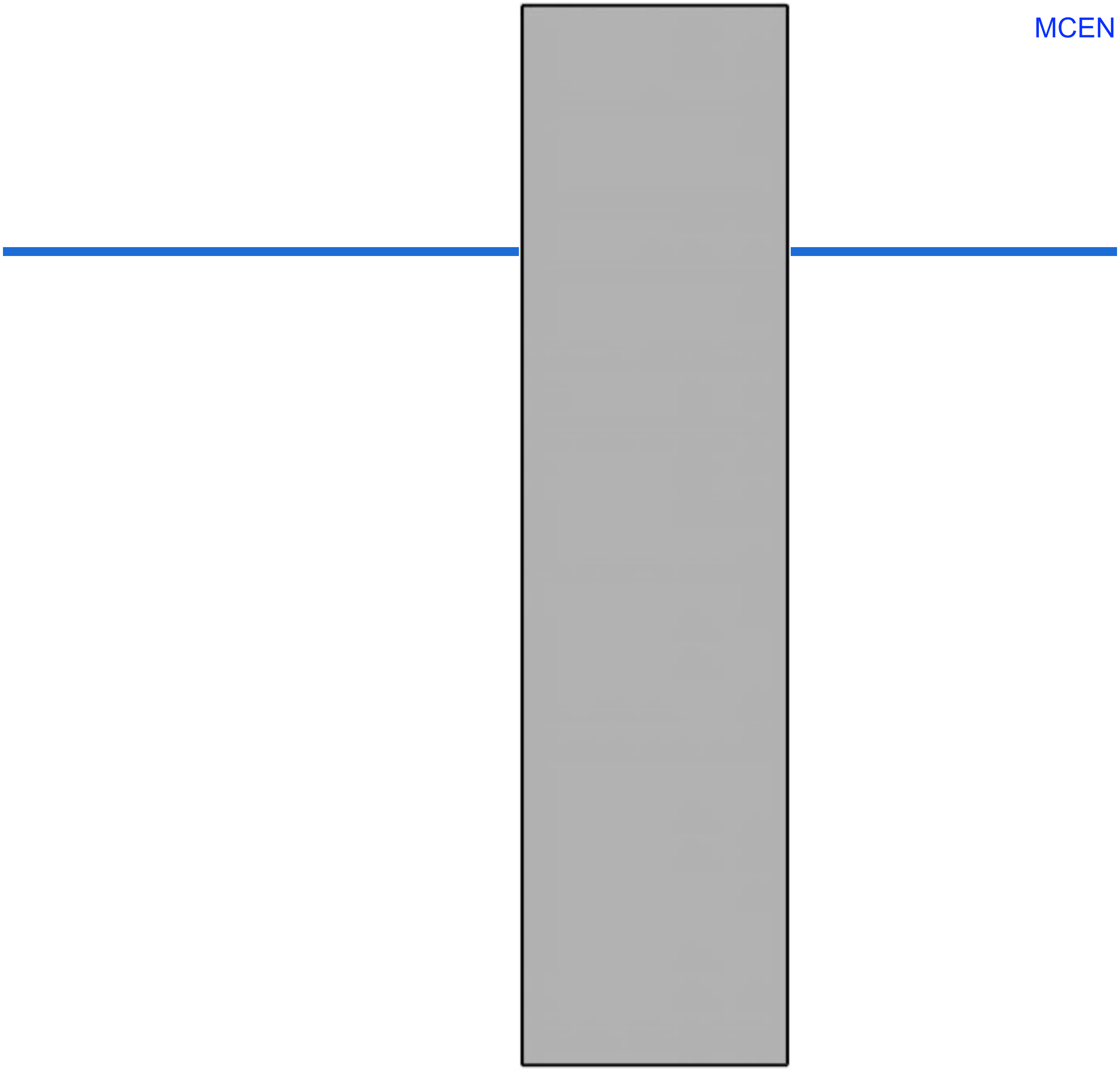




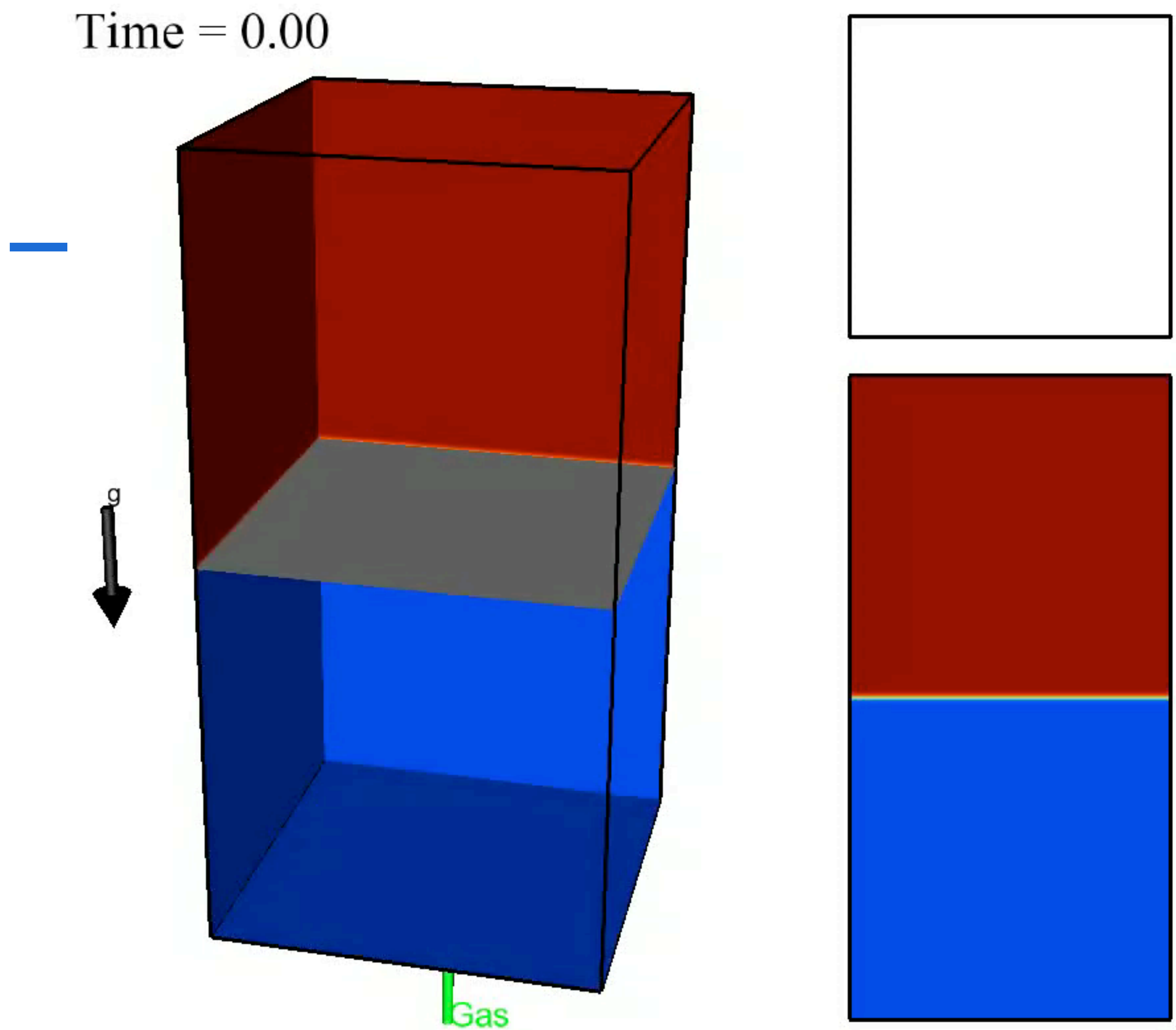

Time [s] = 0.042001
Axial Velocity [m/s]
-40
-10
20
50
80

Turbulence
Desjardins

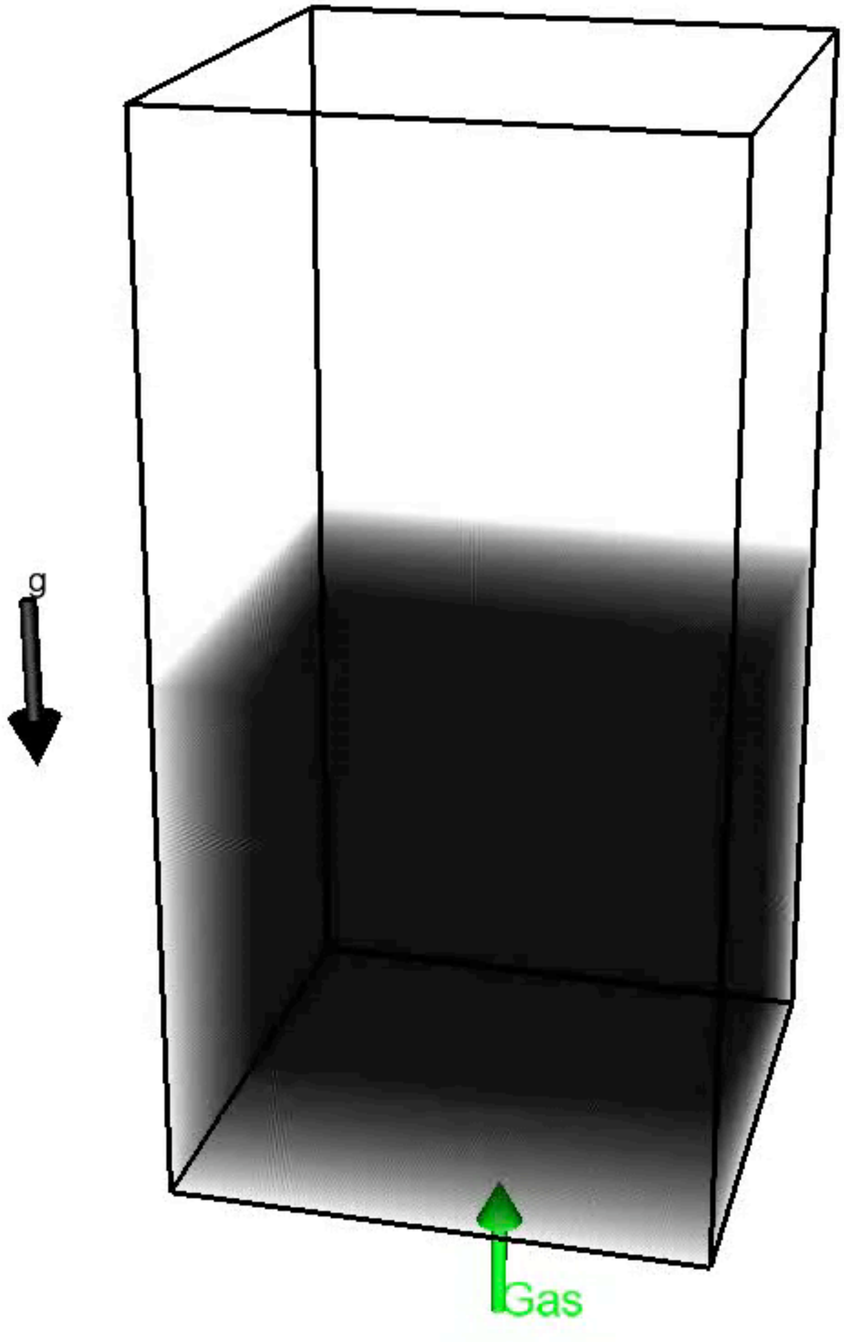


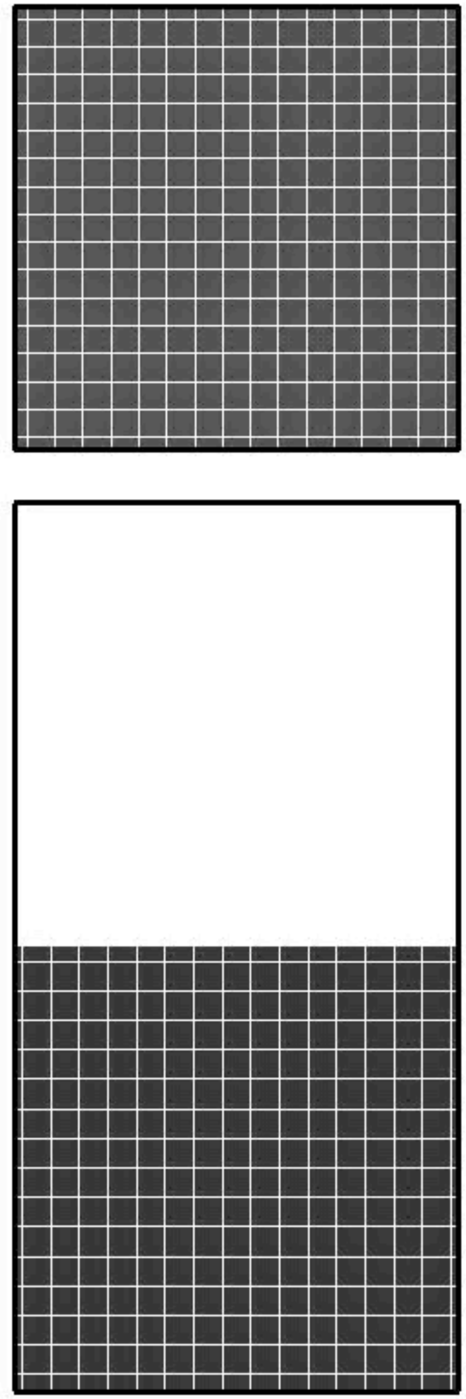


rbulence
esjardins

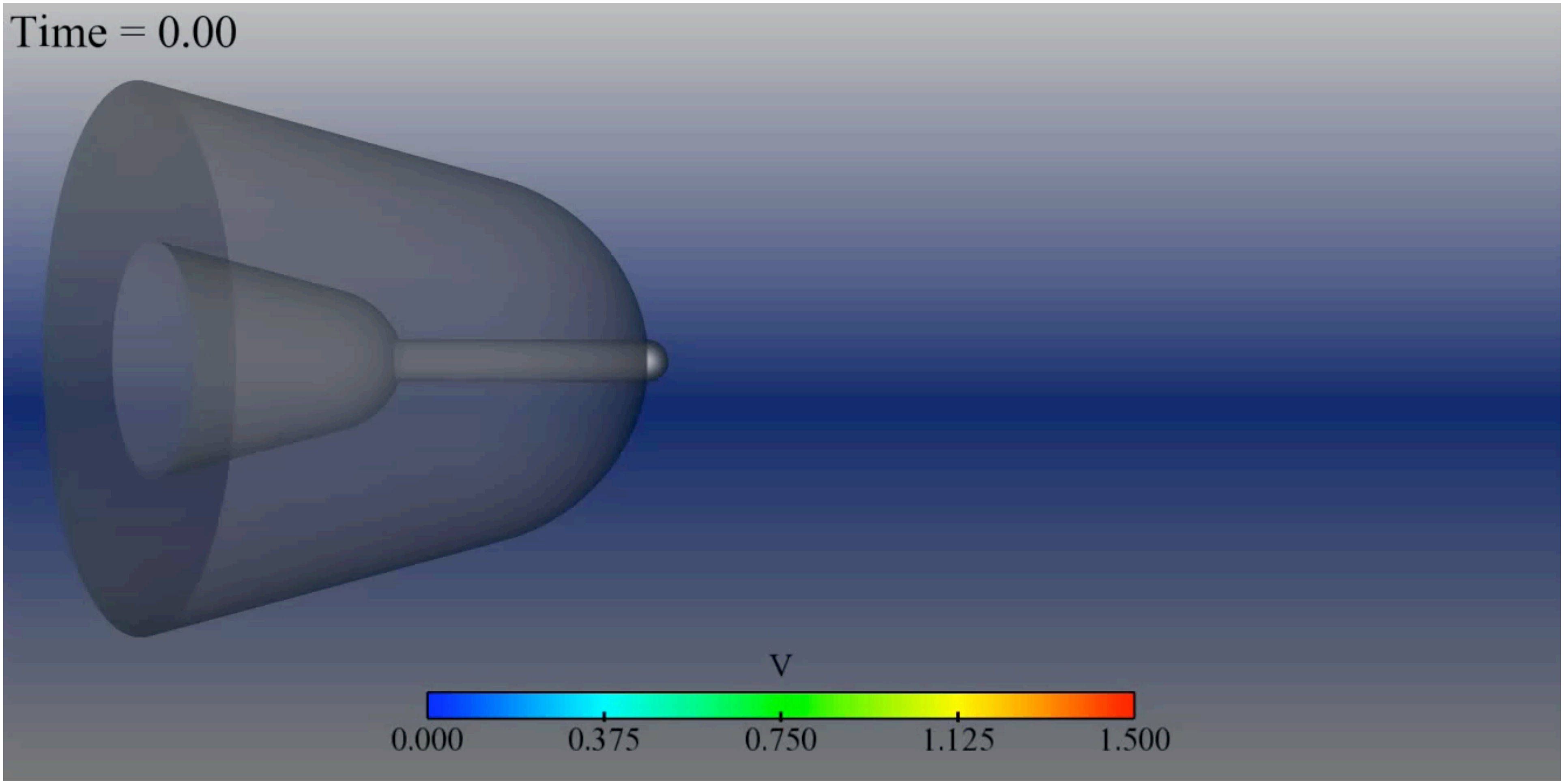


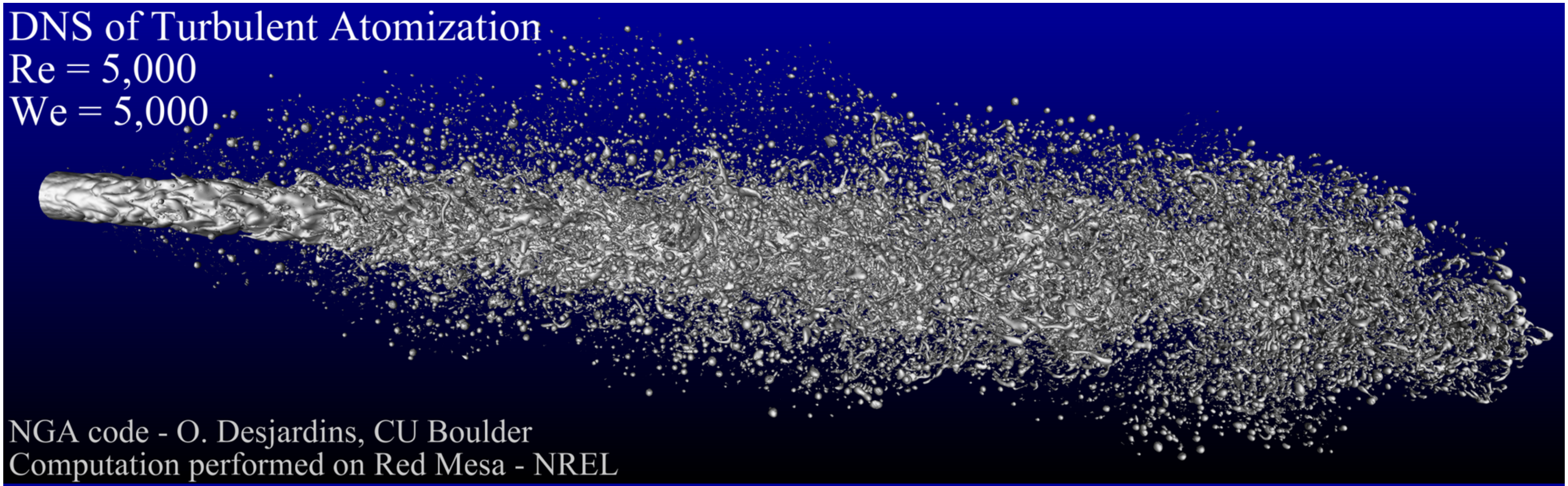
Time = 0.00

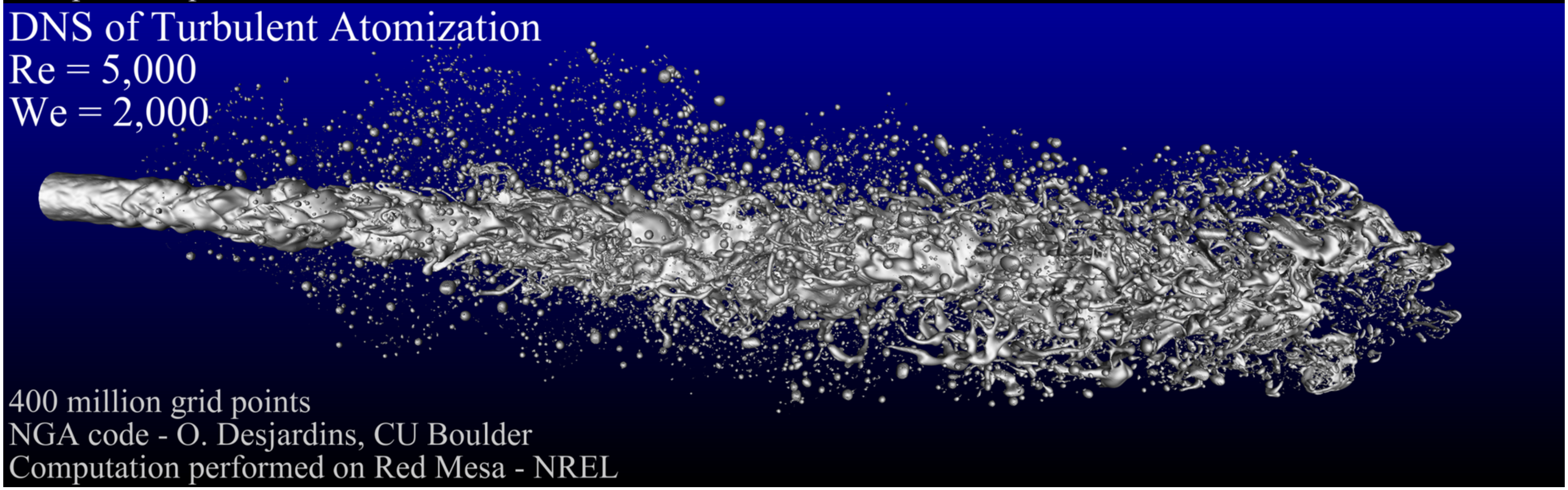




rbulence
esjardins

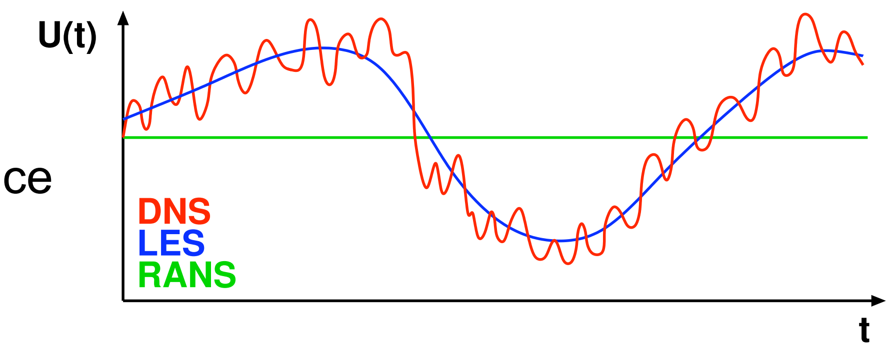

Time = 0.00
V
0.000
0.375
0.750
1.125
1.500


DNS of Turbulent Atomization
Re = 5,000
We = 5,000
NGA code - O. Desjardins, CU Boulder
Computation performed on Red Mesa - NREL

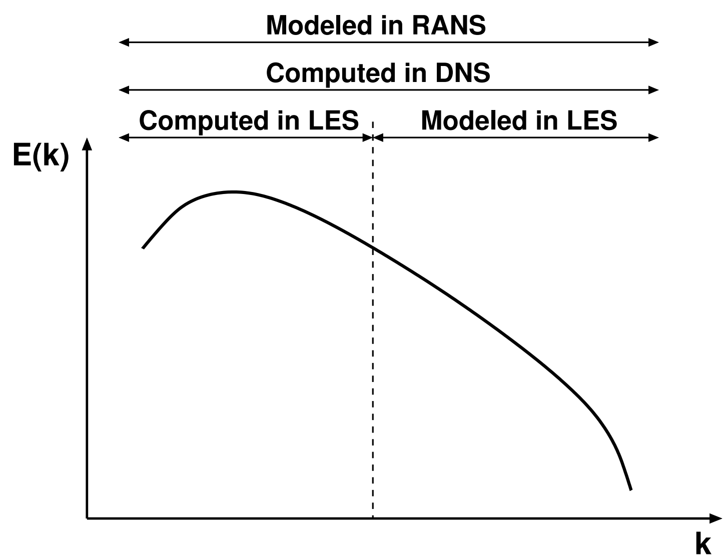

DNS of Turbulent Atomization
Re = 5,000
We = 2,000
400 million grid points
NGA code - O. Desjardins, CU Boulder
Computation performed on Red Mesa - NREL

Turbulence Modeling

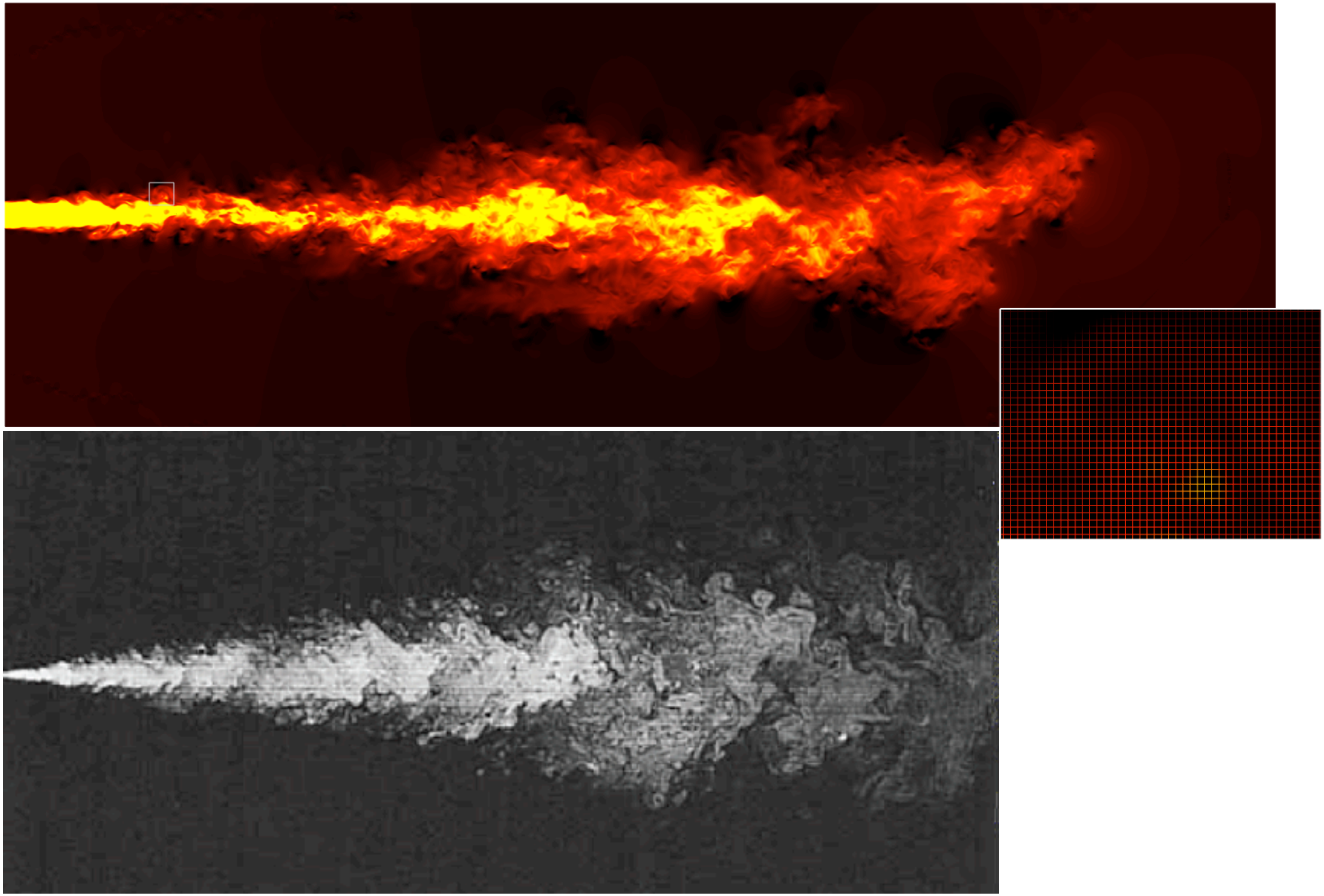
– Direct Numerical Simulation (DNS)
  - Resolve all flow scales directly
  - No need for physical model of turbulence
  - Very high cost



– Large Eddy Simulation (LES)
  - Resolve large flow scales
  - Model small scales only
  - Moderate cost



– Reynolds Averaged Navier-Stokes (RANS)
  - Solve for mean flow
  - Model all fluctuations
  - Low cost



Turbulence Modeling

Turbulent Viscosity Models

Model turbulent transport similarly to molecular transport.

$$\langle u_i u_j \rangle = \frac{2}{3} k \delta_{ij} - \nu_t \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right)$$

Two assumptions:

1. Reynolds stress depends on velocity gradient
2. Particular form of model (directional dependence)

Because of the long time scales
$\Rightarrow$ Turbulence does not align rapidly with mean strain rate.

Turbulence Modeling

Mixing length model for ν_t (Prandtl, 1925)

$$\nu_t \sim l^* u^*,$$

$$l^* = l_m$$

Fluctuation v transports fluid of velocity $\langle U \rangle$ form x_0 to $x_0 + l_m$

$$u^* = u(x_0 + l_m) \approx \langle U \rangle (x + l_m) - \langle U(x) \rangle = \frac{\partial \langle U \rangle}{\partial y} l_m + \cdots$$

$$\Rightarrow \quad \nu_t = l_m^2 \left| \frac{\partial \langle U \rangle}{\partial y} \right|$$

- l_m has to be specified
- For $\frac{\partial \langle U \rangle}{\partial y} = 0, \Rightarrow \nu_t = 0$, incorrect, e.g. turbulent jet centerline

Turbulence Modeling

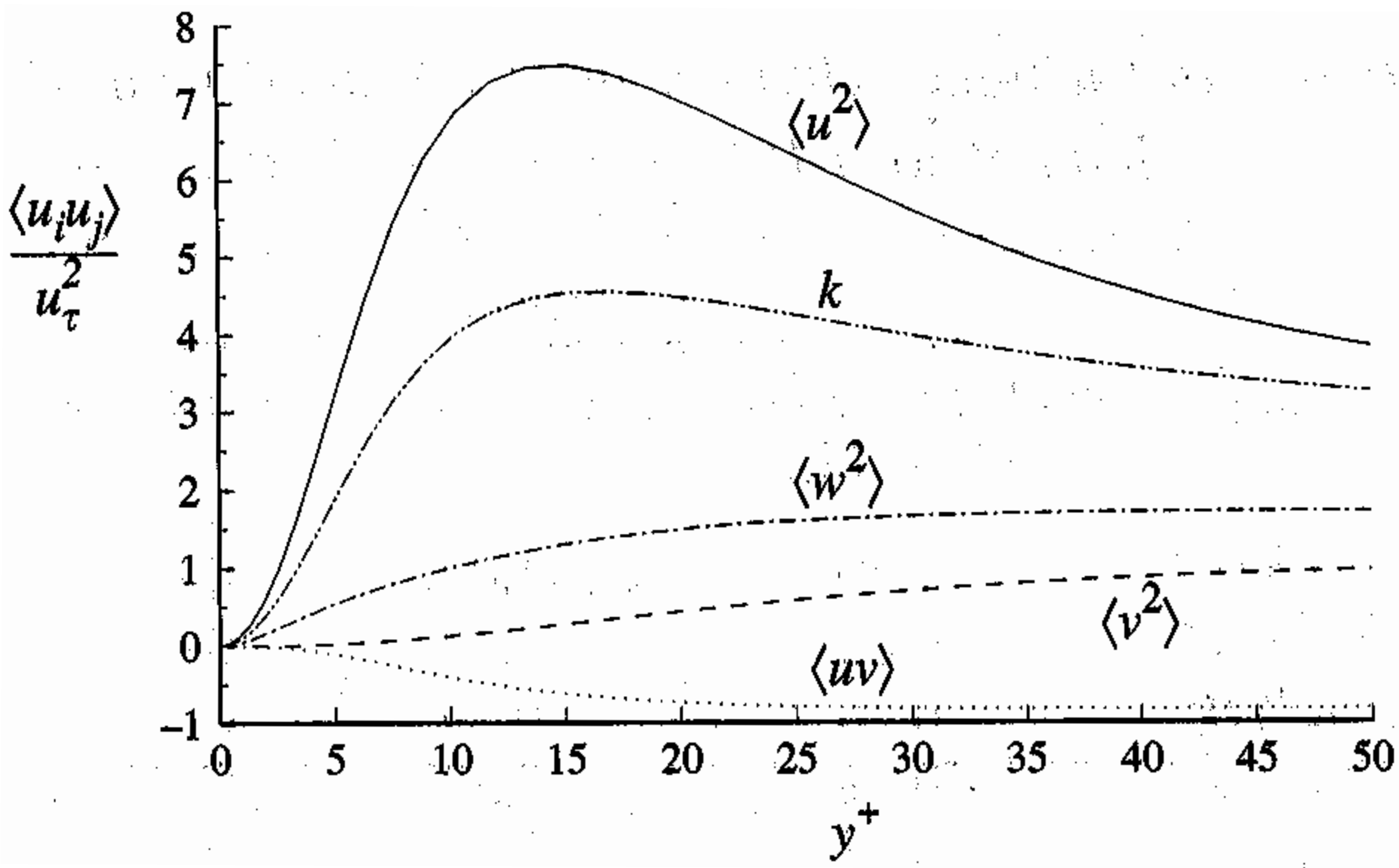
Turbulent Kinetic Energy Models (Prandtl, 1945)

- u^* from TKE: $u^* = ck^{1/2}$

$$\Rightarrow \nu_t = c l_m k^{1/2}; \qquad c \approx 0.55$$

- k from modeled transport equation
- l_m still has to be specified
- Once l_m is specified
  $\Rightarrow$ Solution of Reynolds equation requires model for Reynolds stress tensor $\langle u_i u_j \rangle$
  $\Rightarrow$ Gradient transport model requires model for ν_t
  $\Rightarrow$ Turbulent kinetic energy model for ν_t requires model for k
  $\Rightarrow$ Solution of TKE equation
  $\Rightarrow$ TKE equation needs to be modeled

Turbulence Modeling



Turbulence Modeling

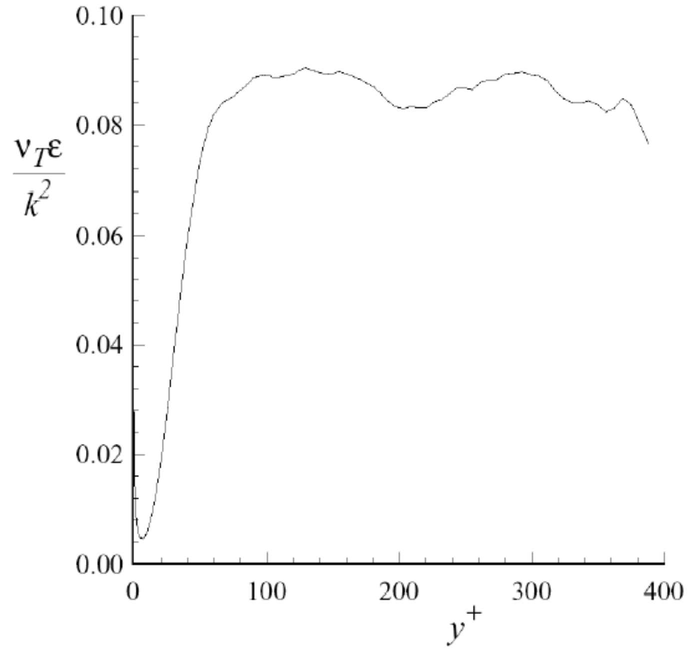


Figure 10.3: Profile of $\nu_T\varepsilon/k^2$ (see Eq. 10.39) from DNS of channel flow at Re= 13,750 (Kim *et al.* (1987)).

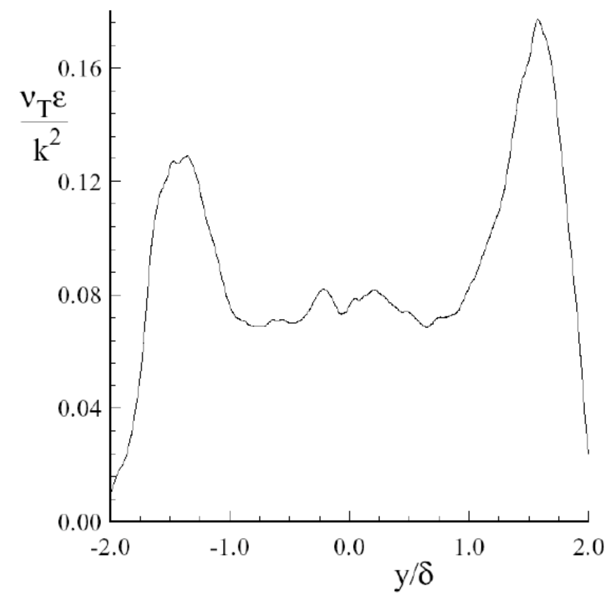


Figure 10.4: Profile of $\nu_T\varepsilon/k^2$ (see Eq. 10.39) from DNS of the temporal mixing layer (from data of Rogers and Moser 1994).

Good agreement with DNS of channel flow (away from walls) and temporal mixing layer (away from edges)

$$\Rightarrow \quad C_\mu = cC_D \approx 0.09$$

Turbulence Modeling

Two equation models: The $k - \varepsilon$ model

Need second equation to determine l_m

e.g. equation for l_m from equation for two point correlaton.

Turbulence Modeling

ε-equation from

- Definition of ε
- Length scale equation and $\varepsilon = C_D k^{3/2}/l_m$
- Empirical

Turbulence Modeling

Other Turbulent Viscosity Models

$k - \omega$ Model

- Turbulence frequency defined as inverse turbulent time-scale

$$\omega \equiv \frac{\varepsilon}{k}$$

- Model equation for ω

$$\frac{\partial \omega}{\partial t} + \langle U_i \rangle \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial x_i} \right) + C_{\omega 1} \frac{\mathcal{P} \omega}{k} - C_{\omega 2} \omega^2$$

Turbulence Modeling

$$\frac{\partial \omega}{\partial t} + \langle U_i \rangle \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\omega}\frac{\partial \omega}{\partial x_i}\right) + (C_{\varepsilon 1} - 1)\frac{\mathcal{P}\omega}{k} - (C_{\varepsilon 2} - 1)\omega^2 + \frac{2\nu_t}{\sigma_\omega k}\nabla\omega \cdot \nabla k$$

Same for homogeneous turbulence, if constants chosen appropriately

But additional term otherwise

In boundary layer flows k-ω- model more accurate for

- Viscous near-wall region
- Presence of streamwise pressure gradient

Turbulence Modeling

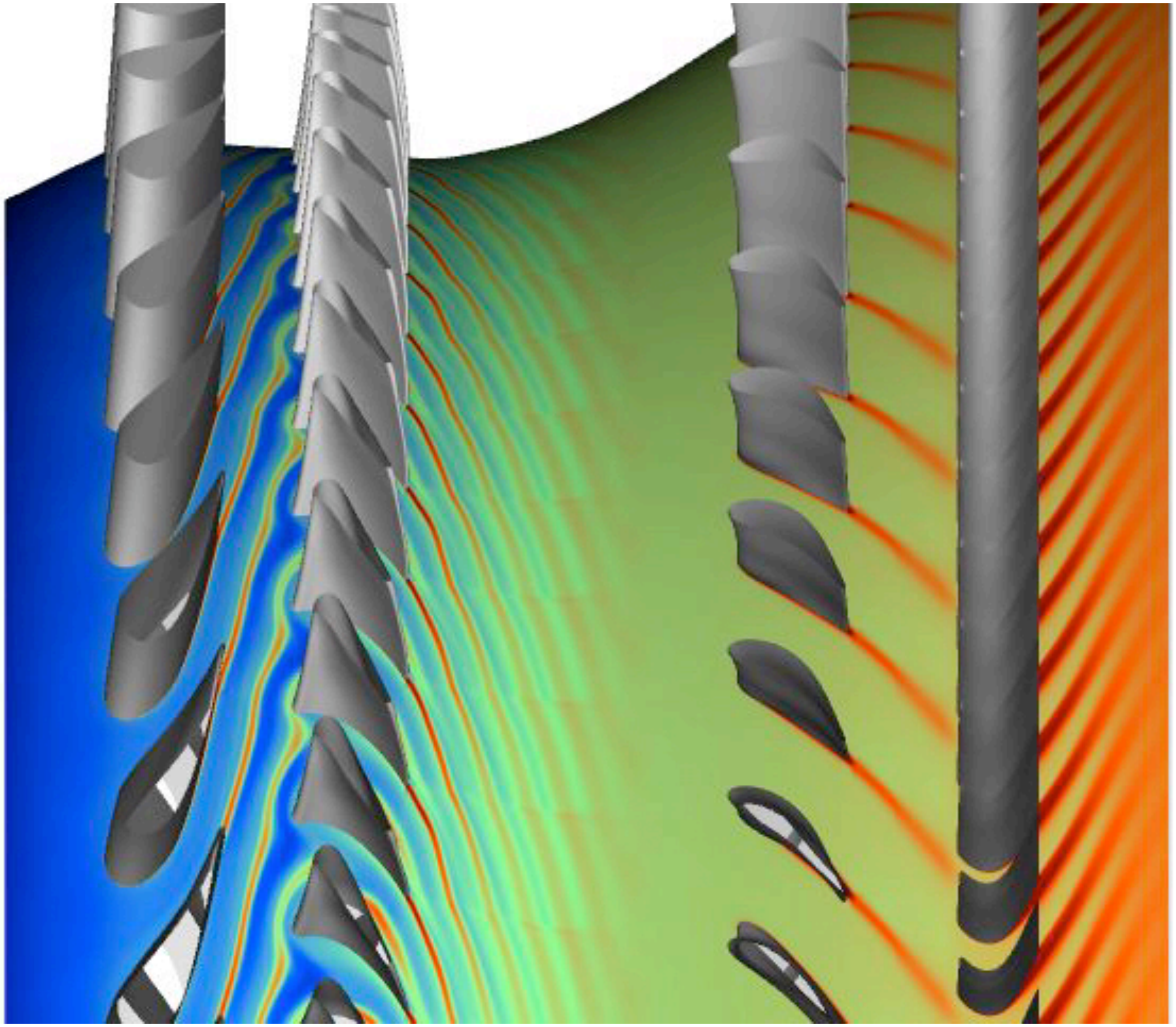
Spalart-Allmaras Model

- ν_t-equation for aerodynamic flows
- Model equation for ν_t

$$\frac{\partial \nu_t}{\partial t} + \langle U_i \rangle \frac{\partial \nu_t}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\nu}\frac{\partial \nu_t}{\partial x_i}\right) + S_\nu$$

- S_ν provided as function of ν, ν_t, vorticity, $|\nabla \nu_t|$, wall distance l_w
- Quite accurate for intended use, but specialized, not general!

Turbulence Modeling



Turbulence Simulation

Direct Numerical Simulation (DNS)

- Solve NS-equations.
- No models
- For turbulent flow
  - Computational domain has to be at least of order of integral length scale l
  - Mesh spacing has to resolve smallest scales η

    $\longrightarrow$ Minimum number of cells per direction $n_x = l/\eta = Re_t^{3/4}$

    $\longrightarrow$ Minimum number of cells total $n_t = n_x^3 = Re_t^{9/4}$

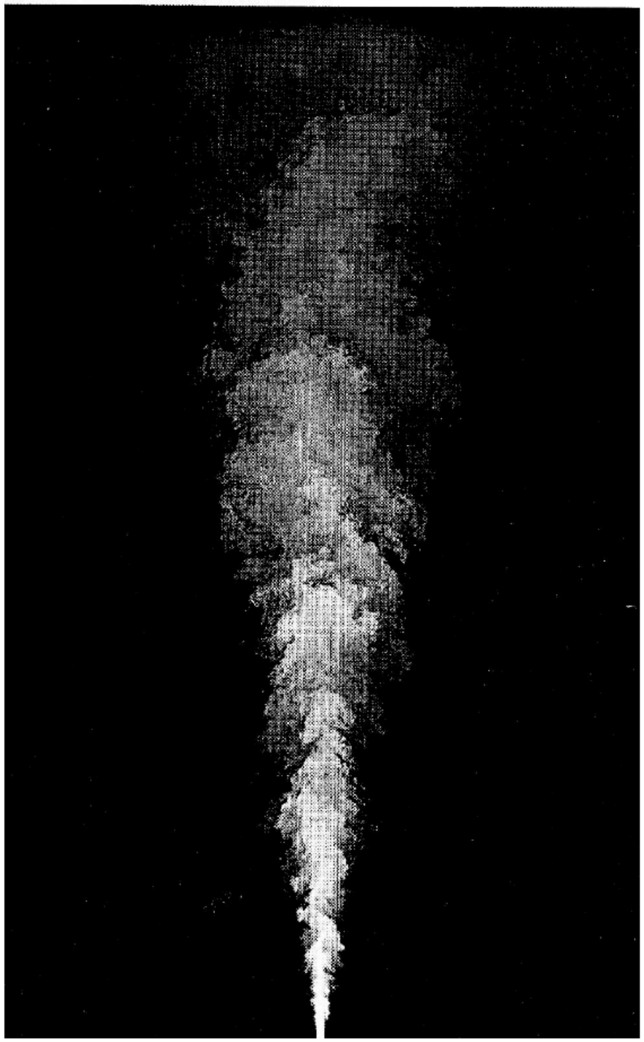
Turbulence Simulation

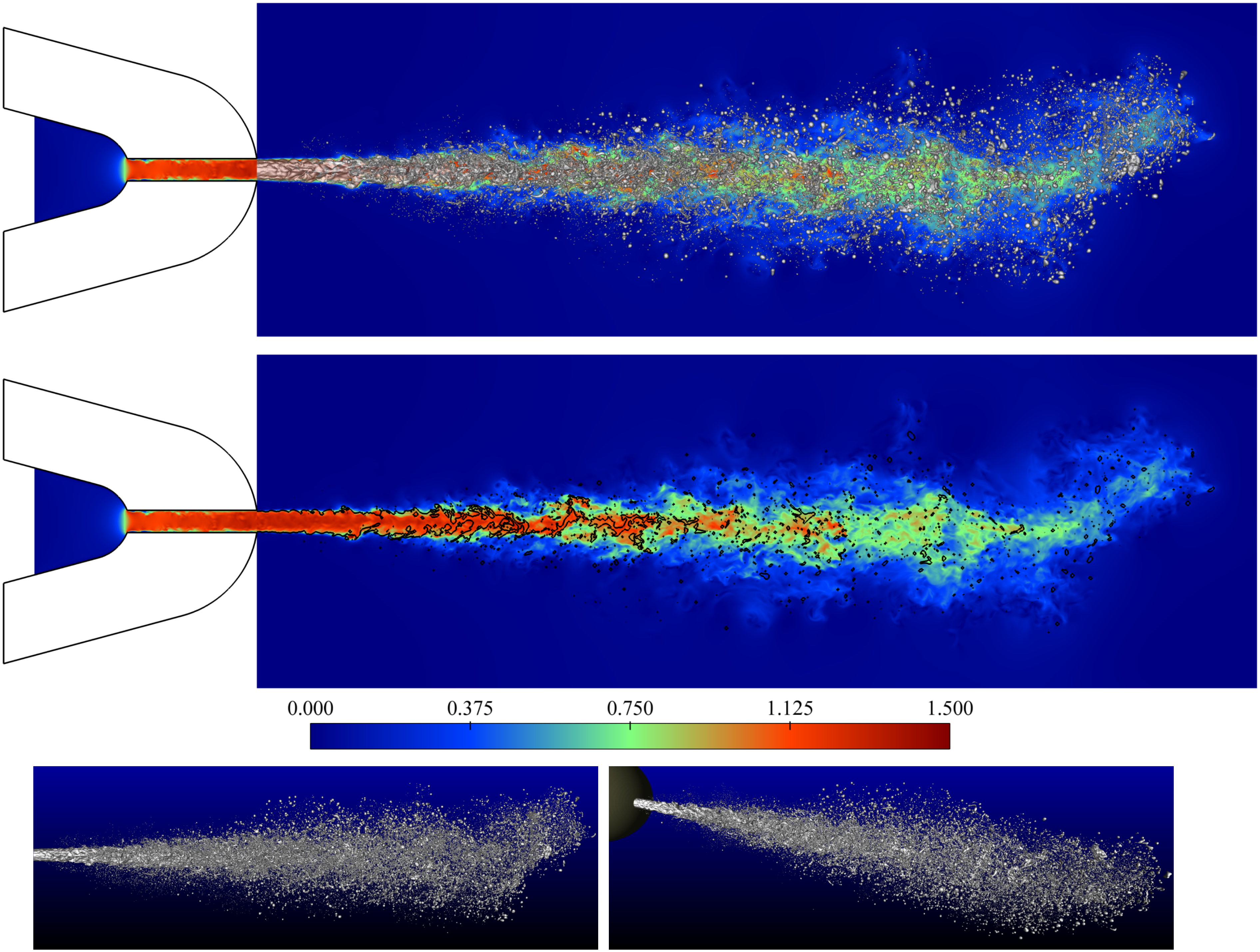
Example: Turbulent Jet with Re = 15000

$$\mathrm{Re}_t = \mathrm{Re}/50$$

$$n_t = \left(\frac{\mathrm{Re}}{50}\right)^{9/4} \approx 375000$$

This is for one integral length scale only!




0.000
0.375
0.750
1.125
1.500

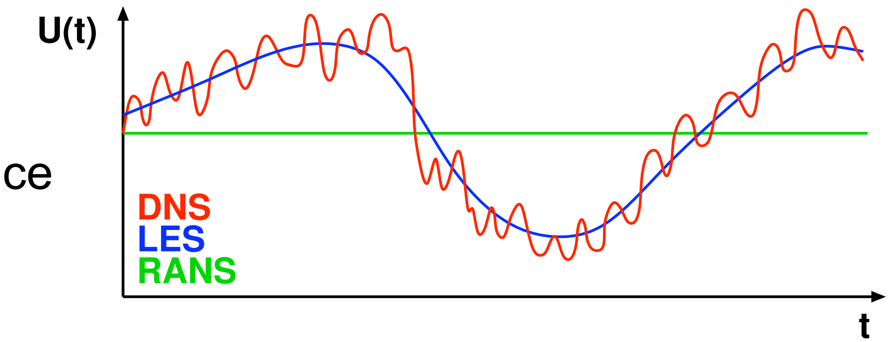
Large-Eddy Simulation

Large-Eddy Simulation (LES)

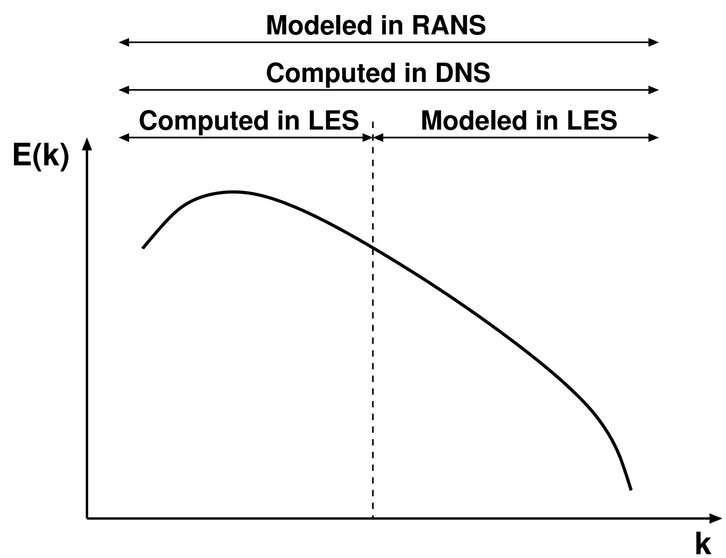
- Spatial filtering as opposed to RANS-ensemble averaging
- Sub-filter modeling as opposed to DNS

Turbulence Modeling

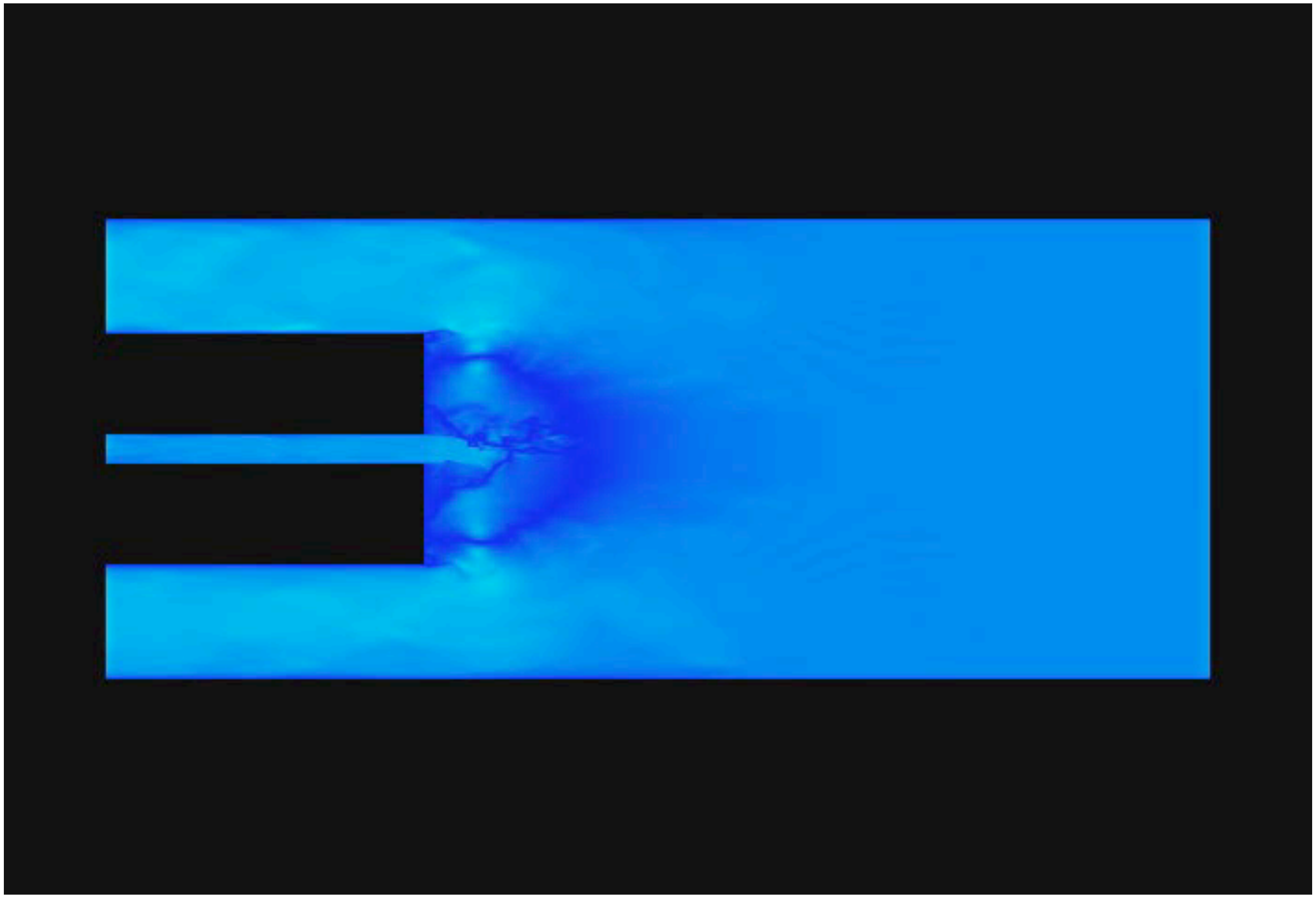
- Direct Numerical Simulation (DNS)
  - Resolve all flow scales directly
  - No need for physical model of turbulence
  - Very high cost

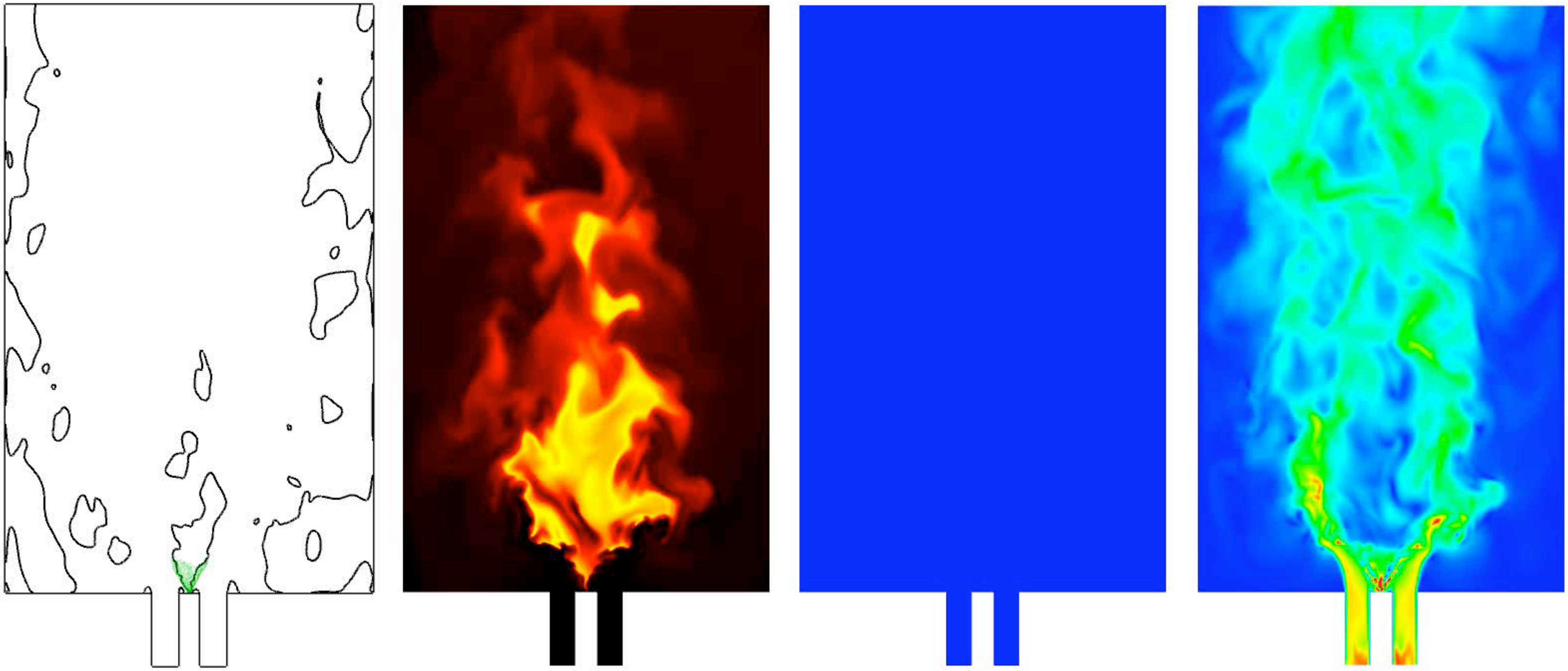


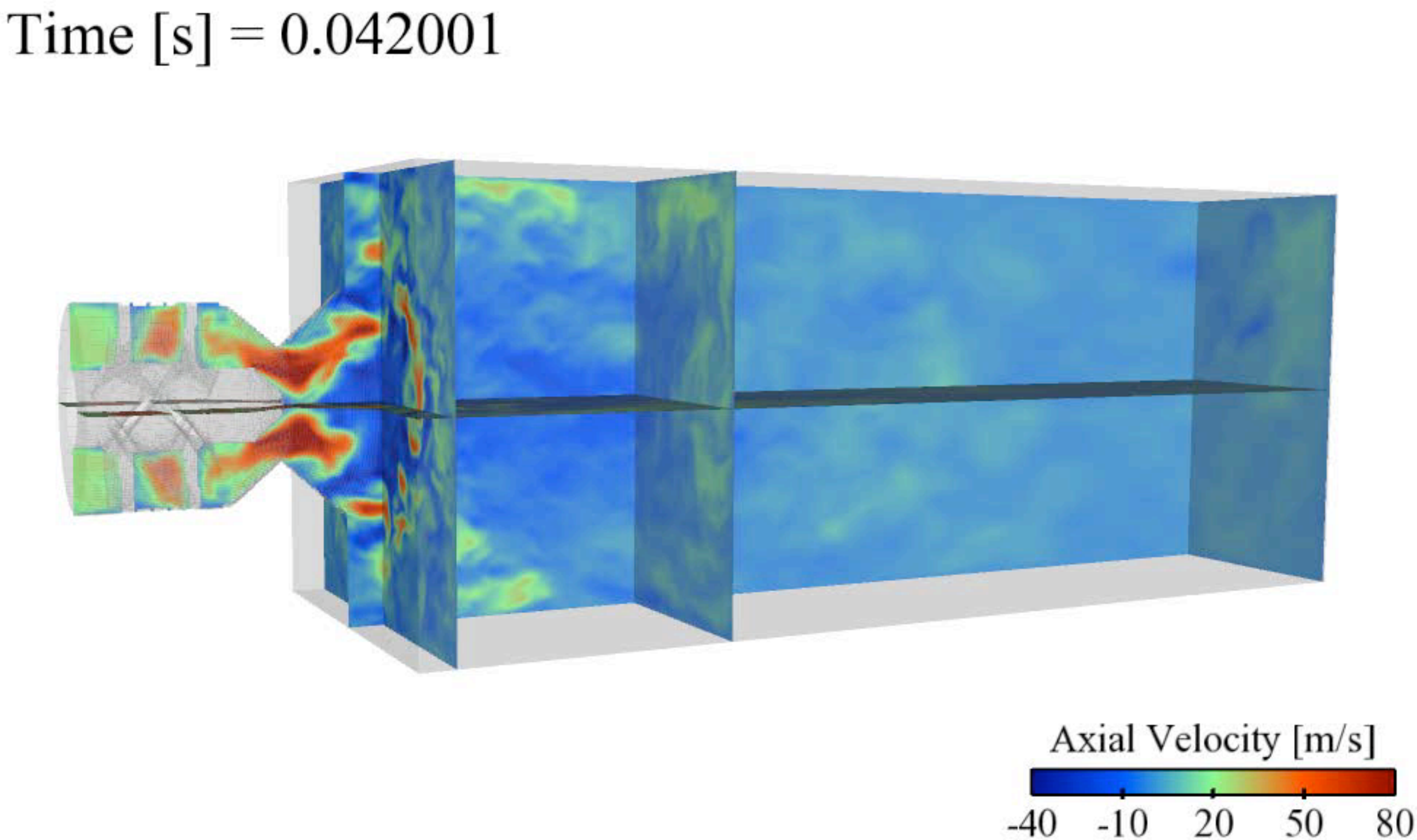
- Large Eddy Simulation (LES)
  - Resolve large flow scales
  - Model small scales only
  - Moderate cost



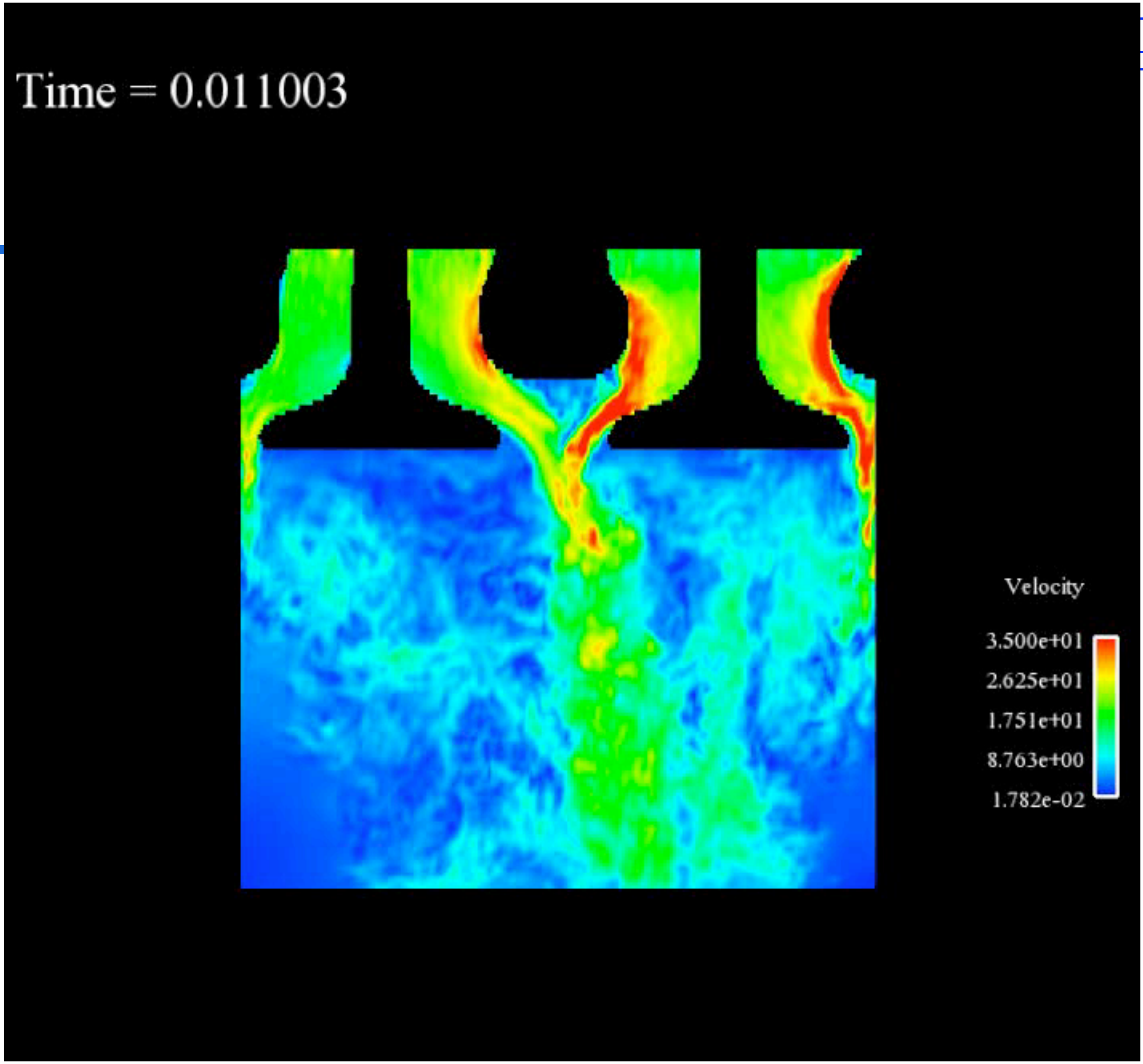
- Reynolds Averaged Navier-Stokes (RANS)
  - Solve for mean flow
  - Model all fluctuations
  - Low cost






Time [s] = 0.042001
Axial Velocity [m/s]
-40
-10
20
50
80

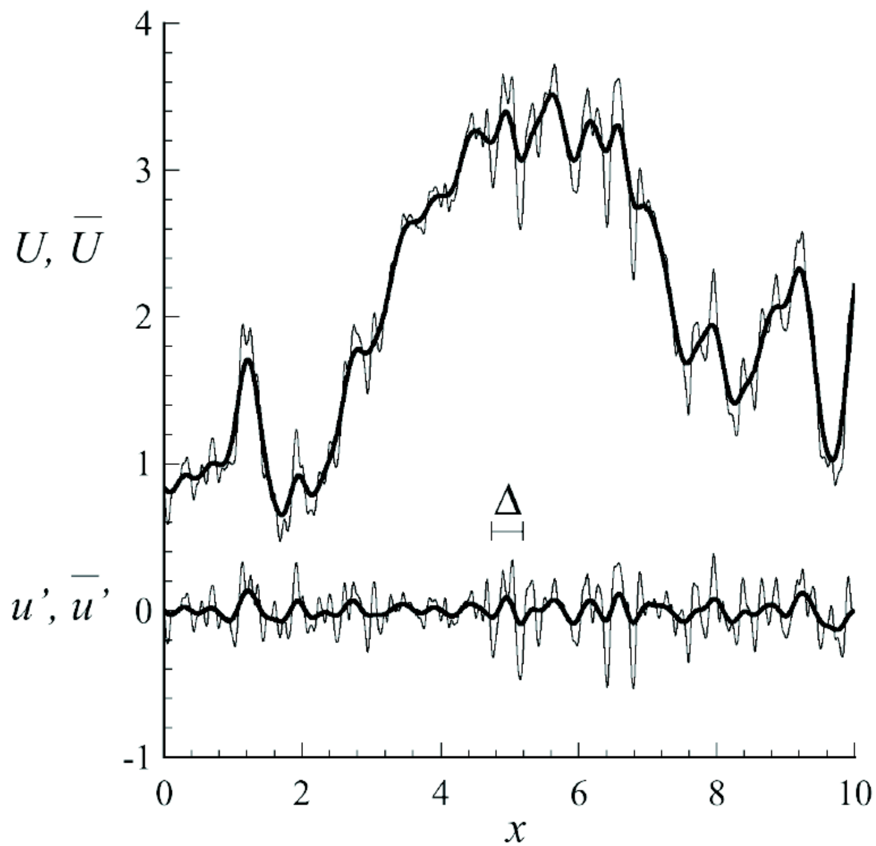
Turbulence
Desjardins



Large-Eddy Simulation

Filtering

At each point in space, the filtered value can loosely be defined as the average value within a box of width Δ



Large-Eddy Simulation

What should be accomplished?

- All scales smaller than a certain length scale should be absent from the filtered quantities
- The filtered signal can then be discretized using a mesh substantially smaller than the DNS mesh

Large-Eddy Simulation

Definition of the filtering procedure:

$$\overline{\boldsymbol{U}}(\boldsymbol{x}) \equiv \int G(\boldsymbol{r}, \boldsymbol{x}) \boldsymbol{U}(\boldsymbol{x} - \boldsymbol{r}, t) d\boldsymbol{r}$$

with

$$\int G(\boldsymbol{r}, \boldsymbol{x}) d\boldsymbol{r} = 1 \,.$$

Residual field

$$\boldsymbol{u}'(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x}) - \overline{\boldsymbol{U}}(\boldsymbol{x})$$

Large-Eddy Simulation

For example:

Box filter in 1D:

$$G(r) = \frac{1}{\Delta} H\left(\frac{1}{2}\Delta - |r|\right) = \begin{cases} 1/\Delta & \text{if } |r| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Sharp spectral filter:

$$\hat{G}(\kappa) = H\left(\frac{\pi}{\Delta} - |\kappa|\right)$$

Large-Eddy Simulation

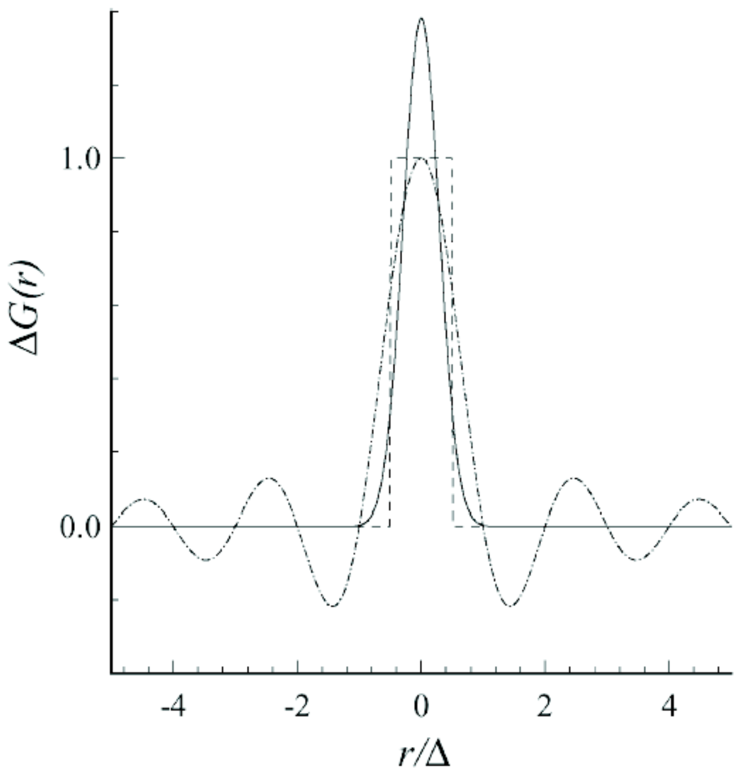


Figure 13.1: Filters $G(r)$: box filter, dashed line; Gaussian filter, solid line; sharp spectral, dot-dashed line.

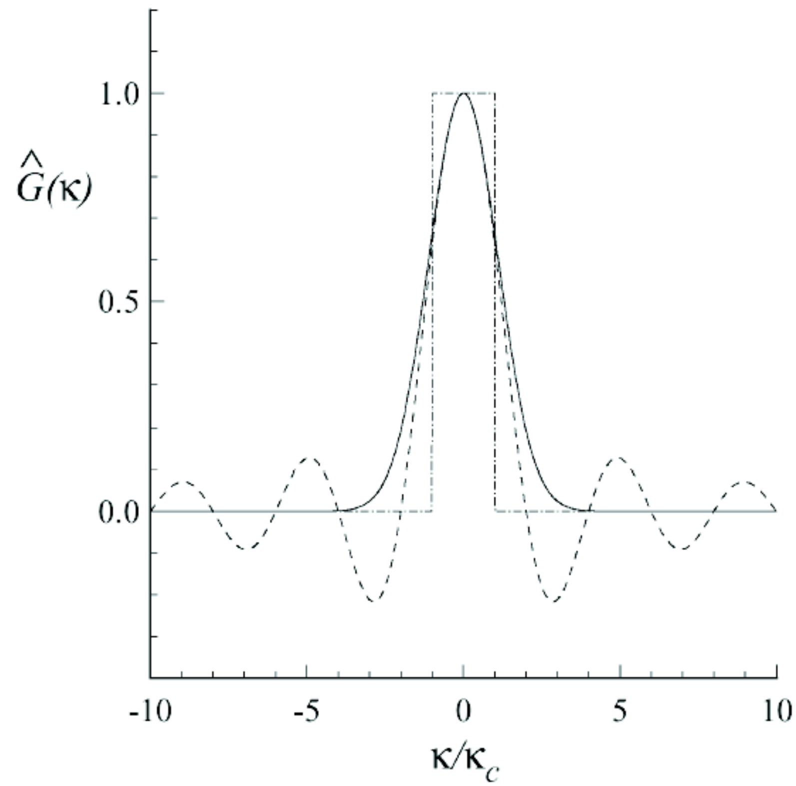


Figure 13.3: Filter transfer functions $\widehat{G}(\kappa)$: box filter, dashed line; Gaussian filter, solid line; sharp spectral filter, dot-dashed line.

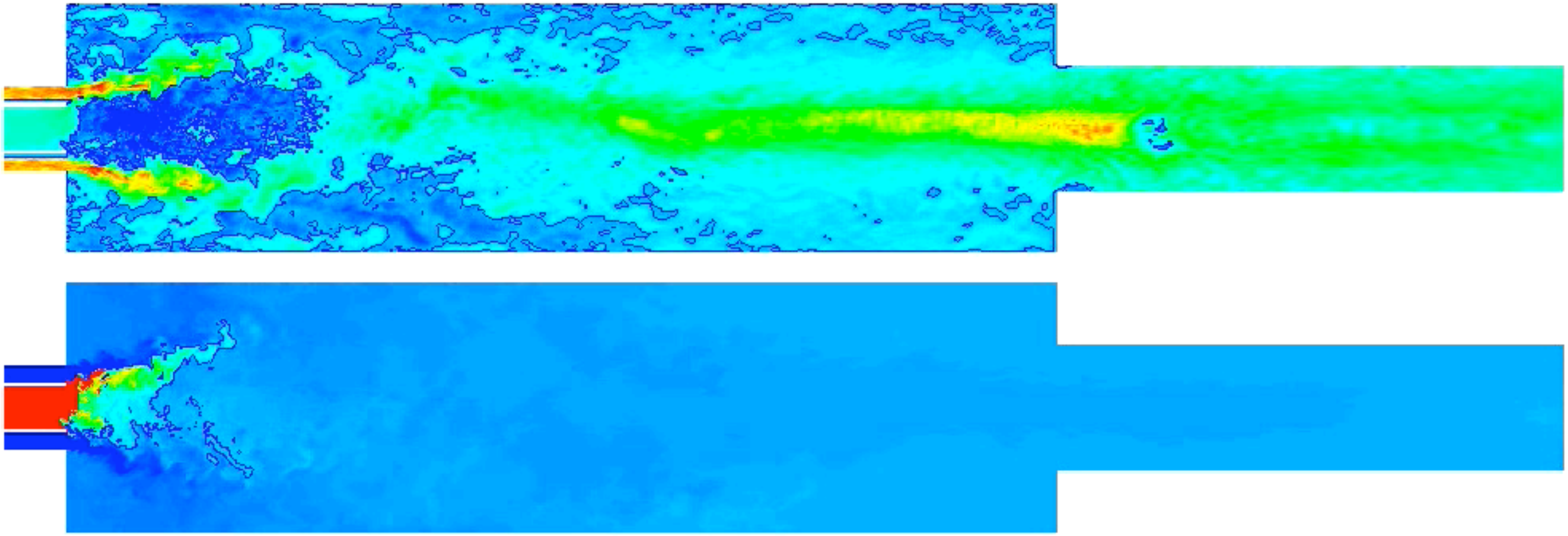
Large-Eddy Simulation

Consequences of filtering

- $\overline{\boldsymbol{U}}(\boldsymbol{x})$ is random quantity
- $\overline{\boldsymbol{U}}(\boldsymbol{x})$ describes large scale motion of turbulence
- TKE largely resolved
- $\overline{\boldsymbol{u}'} \neq 0$

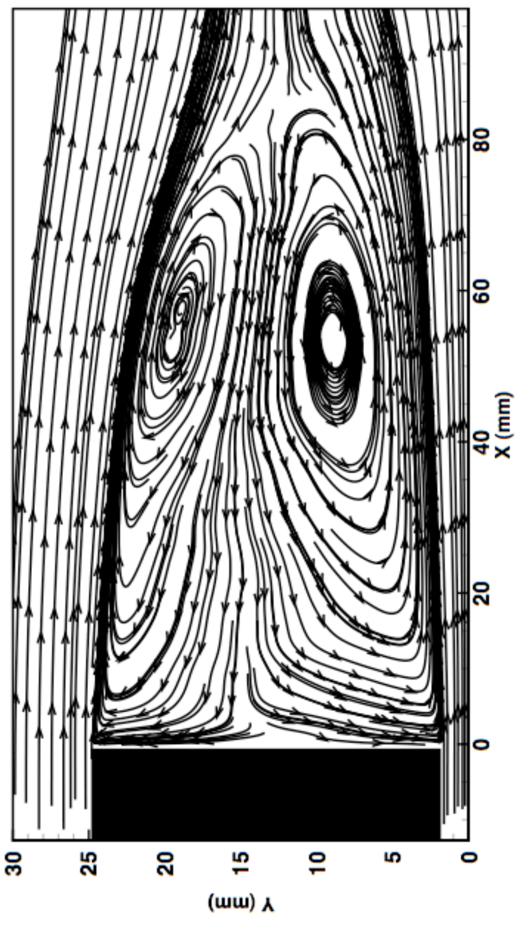
Large-Eddy Simulation

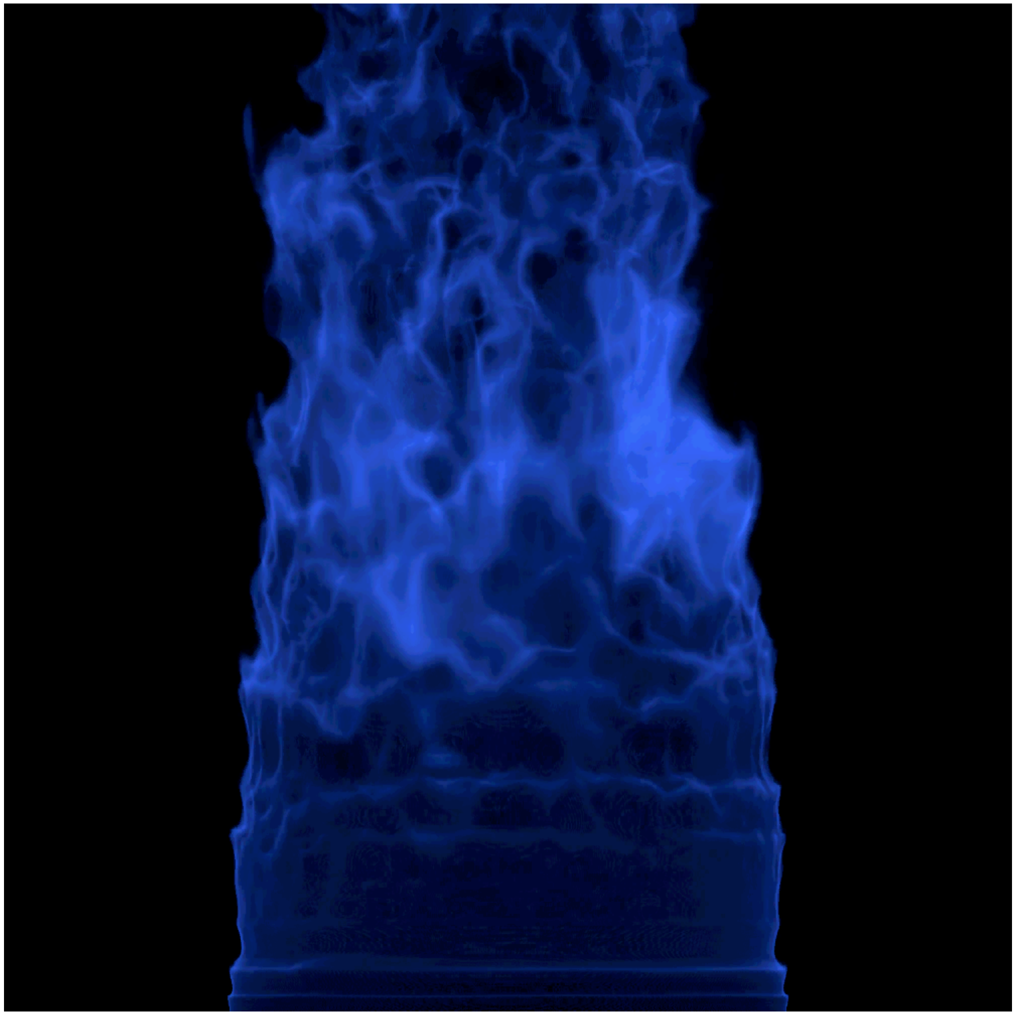
Example:

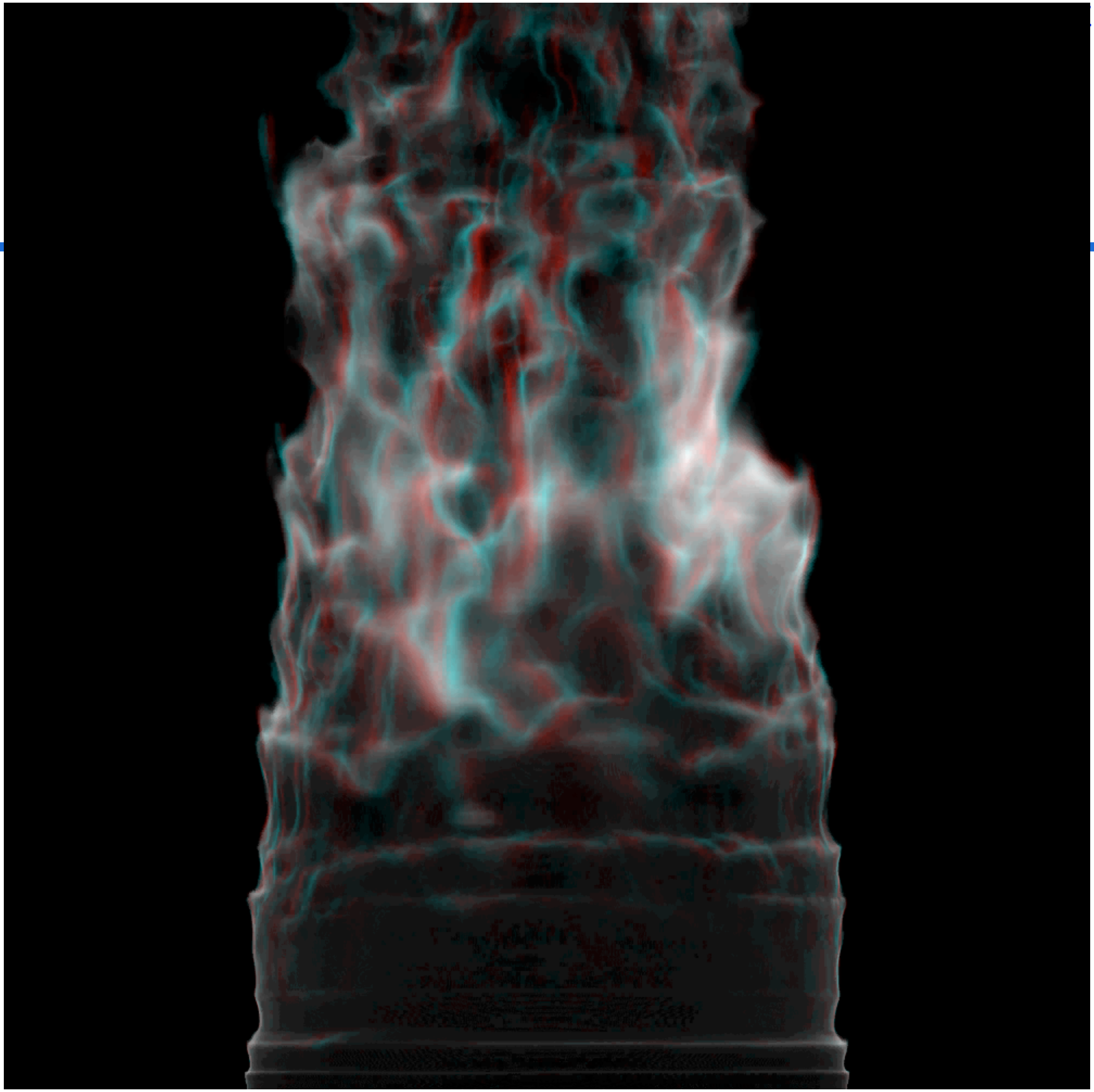


Large-Eddy Simulation

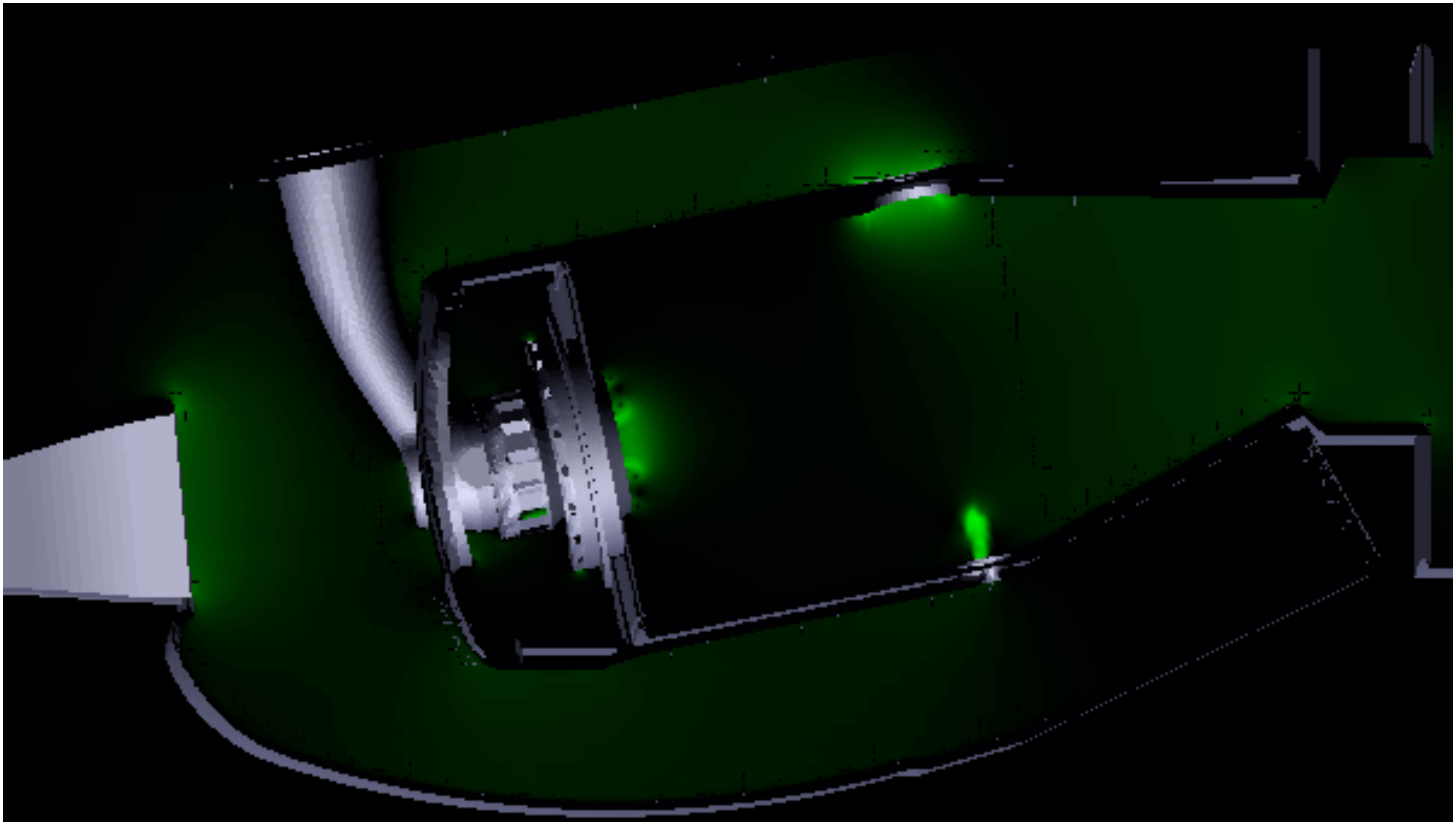
Example:





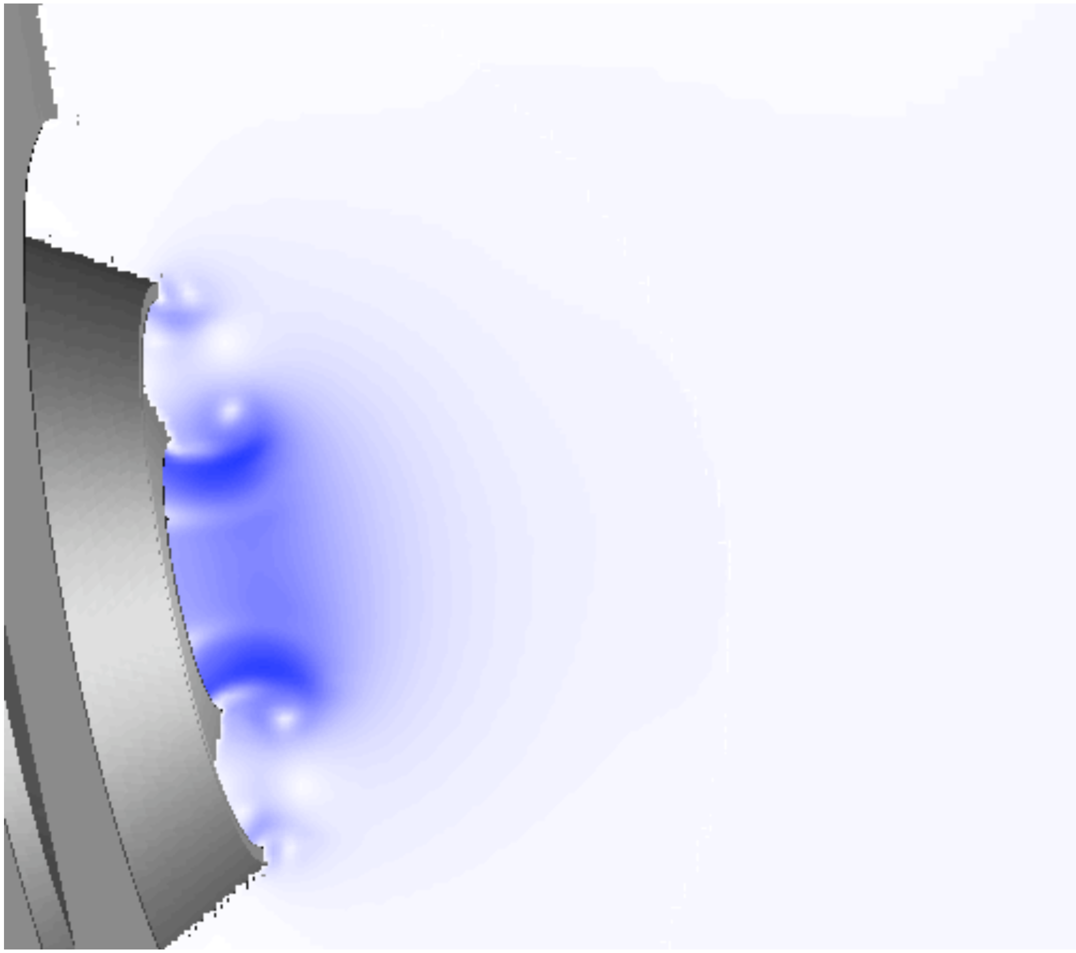


Large-Eddy Simulation



Large-Eddy Simulation

Example:



Large-Eddy Simulation

Properties of the filter

- Filter function independent of time

  $\longrightarrow$ Filtering commutes with time derivative

- For spatially inhomogeneous filter

  $\longrightarrow$ Filtering does not commute with spatial derivatives

Large-Eddy Simulation

Filtered Conservation Equations

(for uniform filters)

Filtered continuity equation:

$$\overline{\frac{\partial U_i}{\partial x_i}} = \frac{\partial \overline{U}_i}{\partial x_i} = 0$$

$$\Rightarrow \qquad \frac{\partial u_i'}{\partial x_i} = \frac{\partial}{\partial x_i}\left(U_i - \overline{U}_i\right) = 0$$

$\Rightarrow$ $\overline{\boldsymbol{U}}$ and $\boldsymbol{u}'$ fields are solenoidal

Large-Eddy Simulation

Filtered momentum equation:

$$\frac{\partial \overline{U}_j}{\partial t} + \frac{\partial \overline{U_i U_j}}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_j} + \nu \frac{\partial^2 \overline{U}_j}{\partial x_i^2}$$

Define residual stress tensor:

$$\tau_{ij}^R \equiv \overline{U_i U_j} - \overline{U}_i \overline{U}_j$$

$$\Rightarrow \qquad \frac{\partial \overline{U}_j}{\partial t} + \frac{\partial \overline{U}_i \overline{U}_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_j} + \nu \frac{\partial^2 \overline{U}_j}{\partial x_i^2} - \frac{\partial \tau_{ij}^R}{\partial x_i}$$

Large-Eddy Simulation

Residual kinetic energy:

$$k_r \equiv \frac{1}{2}\tau_{ii}^R$$

Anisotropic residual stress tensor:

$$\tau_{ij}^r = \tau_{ij}^R - \frac{1}{3}\tau_{kk}^R = \tau_{ij}^R - \frac{2}{3}k_r\delta_{ij}$$

Modified pressure:

$$\overline{p}^* \equiv \overline{p} + \frac{2}{3}k_r$$

$$\Rightarrow \qquad \frac{\partial \overline{U}_j}{\partial t} + \frac{\partial \overline{U}_i \overline{U}_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \overline{p}^*}{\partial x_j} + \nu \frac{\partial^2 \overline{U}_j}{\partial x_i^2} - \frac{\partial \tau_{ij}^r}{\partial x_i}$$

Large-Eddy Simulation

Sub-Filter Modeling

Eddy viscosity model for τ^r_{ij}

Filtered rate of strain tensor

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right)$$

$$\tau^r_{ij} = -\nu_r \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right) = -2\nu_r \overline{S}_{ij}$$

Large-Eddy Simulation

Smagorinsky model for ν_r

(in analogy to mixing length model)

Sub-filter eddy viscosity

$$\nu_r = u'_\Delta l_\Delta = u'_\Delta l_S$$

Sub-filter velocity fluctuation

$$u'_\Delta = l_S \overline{\mathcal{S}}$$

with filtered rate of strain

$$\overline{\mathcal{S}} = (2\overline{S}_{ij}\overline{S}_{ij})^{1/2}$$

Large-Eddy Simulation

$$\Rightarrow \qquad \nu_r = l_S^2 \overline{\mathcal{S}}$$

Smagorinsky length scale $\qquad l_S = C_S \Delta$

$$\Rightarrow \qquad \nu_r = (C_S \Delta)^2 \overline{\mathcal{S}}$$

$$\Rightarrow \qquad \tau_{ij}^r = (C_S \Delta)^2 \overline{\mathcal{S}}\, \overline{S}_{ij}$$

Large-Eddy Simulation

Dynamic Model

Germano's idendity:

Residual stress at filter size Δ

$$\tau_{ij}^R \equiv \overline{U_i U_j} - \overline{U}_i \overline{U}_j$$

Filtering τ_{ij}^R with a larger filter of size $\widehat{\Delta} = 2\Delta$ leads to

$$\widehat{\tau}_{ij}^R = \widehat{\overline{U_i U_j}} - \widehat{\overline{U}_i \overline{U}_j}$$

Residual stress from momentum equation derived at larger filter size $\widehat{\Delta}$

$$T_{ij}^R \equiv \widehat{\overline{U_i U_j}} - \widehat{\overline{U}}_i \widehat{\overline{U}}_j$$

Large-Eddy Simulation

From this follows

$$T_{ij}^R - \widehat{\tau}_{ij}^R = \widehat{\overline{U}_i \overline{U}_j} - \widehat{\overline{U}}_i \widehat{\overline{U}}_j \equiv \mathcal{L}_{ij}$$

- Determine RHS from resolved field
- Replace LHS by Smagorinsky model

$$\tau_{ij}^R = -2C\Delta^2 \overline{\mathcal{S}}\,\overline{S}_{ij} + \frac{1}{3}\tau_{ii}^R \delta_{ij}$$

$$T_{ij}^R = -2C\widehat{\Delta}^2 \widehat{\overline{\mathcal{S}}}\,\widehat{\overline{S}}_{ij} + \frac{1}{3}T_{ii}^R \delta_{ij}$$

Large-Eddy Simulation

$\Rightarrow$ Equation for coefficient C

$$\mathcal{L}_{ij} \equiv \widehat{\overline{U}_i \overline{U}_j} - \widehat{\overline{U}}_i \widehat{\overline{U}}_j = 2C\Delta^2 \overline{\mathcal{S}}\,\overline{S}_{ij} - 2C\widehat{\Delta}^2 \widehat{\overline{\mathcal{S}}}\,\widehat{\overline{S}}_{ij} - \frac{1}{3}\tau^R_{ii}\delta_{ij} + \frac{1}{3}T^R_{ii}\delta_{ij} \equiv C\mathcal{M}_{ij} - \frac{1}{3}(\tau^R_{ii} - T^R_{ii})\delta_{ij}$$

Contract with $\overline{S}_{ij}$

$$\mathcal{L}_{ij}\overline{S}_{ij} = C\mathcal{M}_{ij}\overline{S}_{ij}$$

and

$$C = \frac{\mathcal{L}_{ij}\overline{S}_{ij}}{\mathcal{M}_{ij}\overline{S}_{ij}}$$

More accurate contraction proposed by Lilly: Least squares minimization of error with respect to C